On the local convexity of the implied volatility curve in uncorrelated stochastic volatility models

Elisa Alòs*
Dpt. d'Economia i Empresa and
Barcelona Graduate School of Economics
Universitat Pompeu Fabra
c/Ramon Trias Fargas, 25-27
08005 Barcelona, Spain

Jorge A. León†
Control Automático
CINVESTAV-IPN
Apartado Postal 14-740
07000 México, D.F., Mexico

Abstract

In this paper we give an alternative proof of the convexity of the implied volatility curve as a function of the strike, for stochastic volatility models in the uncorrelated case. Our method is based on the computation of the corresponding first and second derivatives, and on Malliavin calculus techniques. We prove that the implied volatility is a locally convex function of the strike, with a minimum at the forward price of the stock, recovering the previous results by Renault and Touzi (1996). Moreover, we obtain an expression for the short-time limit of the smile in terms of the Malliavin derivative of the volatility process. Our analysis only needs some general integrability and regularity conditions in the Malliavin calculus sense and does not need the volatility to be Markovian nor a diffusion process, as we can see in the examples

1 Introduction

It is well-known that stochastic volatility models capture some important features of the implied volatility. For example, its variation with respect to the

*Supported by the grants ECO2011-288755 and MEC FEDER MTM 2009-08869
†Partially supported by the CONACyT grant 220303

strike price, described graphically as a *smile* or *skew*, (see Renault and Touzi (1996) or Alòs, León and Vives (2007)).

This paper is devoted to the analytical study of the convexity of the implied volatility curve as a function of the strike, in the case of uncorrelated stochastic volatility. As proved in Renault and Touzi (1996) (by induction on the number of posible values that the squared future average volatility can take) this function is locally convex with a minimum at the forward stock price.

In this paper we present an alternative proof of this convexity result, based on the computation of the at-the-money first and second derivatives of the implied volatility. Our method uses implicit differentiation and Malliavin calculus techniques, and gives us explicit expressions for these derivatives. These expressions allow us to proof the convexity of the implied volatility curve, as well as to compute its short-time behaviour in terms of the Malliavin derivative of the volatility process. Our analysis only needs some regularity conditions and does not need the volatility to be a Markovian process, as we see in the examples.

The paper is organized as follows. In Section 2 we introduce the framework and the notation that we utilize in this paper. In Section 3 we prove that the implicit volatility has a stationary point at the forward stock price. In Section 4, we obtain an expression for the second derivative that allows us to prove the local convexity of the implied volatility, as well as to compute its short-time at-the-money limit. Finally, some examples are given in Section 5.

2 Statement of the problem and notation

In this paper we consider the following model for the log-price of a stock under a risk-neutral probability measure P:

$$X_t = x + \tilde{r}t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s, \quad t \in [0,T]. \tag{1}$$

Here, x is the current log-price, $\tilde{r}$ is the instantaneous interest rate, W is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{G}, P)$, and σ is a square-integrable and right-continuous stochastic process adapted to the filtration generated by another standard Brownian motion B independent of W. In the following we denote by $\mathcal{F}^W$ and $\mathcal{F}^B$ the filtrations generated by W and B. Moreover we define $\mathcal{F} := \mathcal{F}^W \vee \mathcal{F}^B$.

It is well-known that there is no arbitrage opportunity if we price an European call with strike price K by the formula

$$V_t = e^{-\tilde{r}(T-t)} E_t[(e^{X_T} - K)_+],$$

where E_t is the $\mathcal{F}_t-$conditional expectation with respect to P (i.e., $E_t(X) = E(X|\mathcal{F}_t)$). In the sequel, we make use of the following notation:

- $v_t^2 = \frac{1}{T-t}\int_t^T \sigma_u^2 du$. That is, v_t represents the future average volatility.
- $M_t = E_t\left(\int_0^T \sigma_u^2 du\right)$.

- $BS(t,x,k,\sigma)$ denotes the price of a European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price $K=\exp(k)$ and interest rate $\tilde{r}$. Remember that in this case

$$BS(t,x,k,\sigma) = e^x N(d_+) - e^{k-\tilde{r}(T-t)} N(d_-),$$

where N denotes the cumulative probability function of the standard normal law and

$$d_\pm := \frac{k_t^* - k}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$

with $k_t^* := x + \tilde{r}(T-t)$.

Notice that, as σ is independent to the filtration generated by W,option prices are given by the so-called *Hull and White formula* (see for example Hull and White (1987))

$$V_t = E_t\left(BS(t,X_t,k,v_t)\right), \quad t\in[0,T]. \tag{2}$$

3 The at-the-money implied volatility skew

Let us define the implied volatility $I = I(t,X_t,k)$ as the stochastic process such that $V_t = BS\left(t,X_t,k,I\right)$. Sometimes we use the convention $I = I(t,k)$ in order to simplify the notation. Our first aim is to study the derivative $\frac{\partial I}{\partial k}|_{k=k_t^*}$. More precisely we get the following result:

Proposition 1 *(the implied volatility skew) Consider the model (1). Then, for all* $t\in[0,T]$, $\frac{\partial I}{\partial k}(t,k_t^*) = 0$.

Proof. This proof can be deduced following the ideas of the proof of Proposition 5.1 in Alòs, León and Vives (2007). Namely, by the definition of I, we know

$$\frac{\partial}{\partial k}V_t = \frac{\partial BS}{\partial k}(t,X_t,k,I) + \frac{\partial BS}{\partial \sigma}(t,X_t,k,I)\frac{\partial I}{\partial k}(t,k), \quad t\in[0,T].$$

Hence,

$$\frac{\partial BS}{\partial \sigma}(t,X_t,k,I)\frac{\partial I}{\partial k} = \frac{\partial}{\partial k}V_t - \frac{\partial BS}{\partial k}(t,X_t,k,I), \quad t\in[0,T]. \tag{3}$$

Also, from (2), we know

$$\frac{\partial V_t}{\partial k} = E_t\left[\frac{\partial BS}{\partial k}\left(t,X_t,k,v_t\right)\right].$$

On the other hand, since

$$\begin{aligned}\frac{\partial BS}{\partial k}(t,x,k_t^*,\sigma) &= -e^x N\left(-\frac{\sigma\sqrt{T-s}}{2}\right)\\ &= \frac{1}{2}\left(BS(t,x,k_t^*,\sigma)-e^x\right),\end{aligned}$$

then we obtain

$$\begin{aligned}&\frac{BS}{\partial k}(t,X_t,k_t^*,\sigma)-\frac{\partial BS}{\partial k}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\\ =\;& \frac{1}{2}\left(BS\left(t,X_t,k_t^*,v_t\right)-BS\left(t,X_t,k_t^*,I(t,k_t^*)\right)\right).\end{aligned}$$

Therefore, (3) yields

$$\begin{aligned}&\frac{\partial BS}{\partial \sigma}\left(t,X_t,k_t^*,I(t,k_t^*)\right)\frac{\partial I}{\partial k}\left(t,k_t^*\right)\\ &= E_t\left(\frac{\partial BS}{\partial k}\left(t,X_t,k_t^*,v_t\right)\right)-\frac{\partial BS}{\partial k}\left(t,X_t,k_t^*,I\left(t,k_t^*\right)\right)\\ &= \frac{1}{2}E_t\left(BS\left(t,X_t,k_t^*,v_t\right)-BS\left(t,X_t,k_t^*,I\left(t,k_t^*\right)\right)\right)\\ &= 0,\end{aligned}$$

where the last inequality is due to (2) and the definition of I. Thus, the proof is complete. ■

Remark 2 *The above theorem proves that, fixed $t\in[0,T]$, the implied volatility $I(t,k)$ has stationary point at $k=k_t^*$. Notice that this result is independent of the stochastic volatility model. That is, we do not need it to be a diffusion or a Markov process.*

4 The at-the-money implied volatility smile

Now our purpose in this section is to study the at-the-money second derivative $\frac{\partial^2 I}{\partial k^2}(t,k_t^*)$. We prove that this is positive. Consequently, for every fixed $t\in[0,T]$, the implied volatility $I(t,X_t,k)$ is a locally convex function of k. Moreover, we prove that $\lim_{t\to T}\frac{\partial^2 I}{\partial k^2}(t,k_t^*)$ is well-defined and finite, which is figure it out explicitly.

We assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (2006). In the remaining of this paper $\mathbb{D}_B^{1,2}$ denotes the domain of the Malliavin derivative operator D^B with respect to the Brownian motion B. It is well-known that $\mathbb{D}_B^{1,2}$ is a dense subset of $L^2(\Omega)$ and that D^B is a closed and unbounded operator from $L^2(\Omega)$ to $L^2([0,T]\times\Omega)$. We will use the notation $\mathbb{L}_B^{1,2}=L^2([0,T];\mathbb{D}_B^{1,2})$.

For our purpose, we introduce the following hypotheses:

(H1) σ^2 belongs to $\mathbb{D}_B^{1,2}$, and there exists an adapted process $Y = \{Y_r, r \in [0,T]\} \in L^4(\Omega \times [0,T])$ such that $\left|E_r\left(D_r^B \sigma_u^2\right)\right| \leq Y_r$, for all $t \leq r \leq u \leq T$.

(H2) For every $r \in [0,T]$, there exists an $\mathcal{F}_r^B-$measurable random variable $D_r^+ \sigma_r$ such that

$$\lim_{t \uparrow T} \frac{\int_t^T E_t\left(\sup_{r \leq u \leq T}\left|E_r\left(D_r^B \sigma_u^2 - D_t^+ \sigma_t^2\right)\right|^4\right) dr}{T-t} = 0.$$

(H3) There exist two deterministic, integrable and right continuous functions $\sigma_1, \sigma_2 : [0,T] \to R^+$ such that

$$\sigma_1(t) \leq |\sigma_t| \leq \sigma_2(t), \quad t \in [0,T].$$

Remark 3 *Notice that under (H1), the Clark-Ocone formula gives us that (see, for instance, Nualart (2006))*

$$M_t = M_0 + \int_0^t \left(\int_s^T E_s\left(D_s^B \sigma_r^2\right) dr\right) dB_s, \quad t \in [0,T],$$

with $M_0 = E\left(\int_0^T \sigma_s^2 ds\right)$.

Before stating the main result of this section, we establish the following auxiliary result.

Lemma 4 *Let* $r \in [t,T]$ *and* $\Lambda_r = E_r\left(BS(t, X_t, k_t^*, v_t)\right)$, *then*

$$\exp\left(\frac{BS^{-1}(t, X_t, k_t^*, \Lambda_r)^2 (T-t)}{8}\right) \leq E_r\left(\exp\left(\frac{v_t^2 (T-t)}{8}\right)\right)$$

and

$$\left(BS^{-1}(t, X_t, k_t^*, \Lambda_r)\right)^{-3} \leq E_r\left(\frac{\exp\left(\frac{3v_t^2(T-t)}{8}\right)}{v_t^3}\right).$$

Proof. We first observe that $BS^{-1}(t, X_t, k_t^*, \cdot)$ and $\exp(\cdot)$ are two convex function on R^+. Therefore, Jensen inequality implies

$$\begin{aligned}
&\exp\left(\frac{BS^{-1}(t, X_t, k_t^*, \Lambda_r)^2 (T-t)}{8}\right) \\
&= \exp\left(\frac{BS^{-1}(t, X_t, k_t^*, E_r(BS(t, X_t, k_t^*, v_t)))^2 (T-t)}{8}\right) \\
&\leq \exp\left(E_r\left(\frac{BS^{-1}(t, X_t, k_t^*, BS(t, X_t, k_t^*, v_t))^2 (T-t)}{8}\right)\right) \\
&\leq E_r\left(\exp\left(\frac{v_t^2 (T-t)}{8}\right)\right).
\end{aligned}$$

Similarly, using that $x \mapsto x^{-3}$ is a convex function on R^+ and the Taylor expansion for $BS^{-1}(t, X_t, k_t^*, \cdot)$, we have

$$\begin{aligned}
&\left(BS^{-1}(t, X_t, k_t^*, \Lambda_r)\right)^{-3} \\
&\leq \left(\frac{\sqrt{2\pi}}{\sqrt{T-t}} E_r \left(BS(t, X_t, k_t^*, v_t)\right) e^{-X_{tt}}\right)^{-3} \\
&\leq E_r \left(\left(\frac{\sqrt{2\pi}}{\sqrt{T-t}} BS(t, X_t, k_t^*, v_t) e^{-X_t}\right)^{-3}\right),
\end{aligned}$$

which implies the result due to the mean value theorem. ■

Theorem 5 *Assume that the model (1), and Hypotheses (H1) and (H3) are satisfied. Then* $\frac{\partial^2 I}{\partial k^2}(t, k_t^*) \geq 0$*, for all* $t \in [0, T]$*. Moreover, if Hypothesis (H2) also holds,*

$$\lim_{t \to T} \frac{\partial^2 I}{\partial k^2}(t, k_t^*) = \frac{(D_t^+ \sigma_t^2)^2}{12 \sigma_t^5}.$$

Proof. From the definition of the implied volatility I, we have

$$\begin{aligned}
\frac{\partial^2}{\partial k^2} V_t = &\frac{\partial^2 BS}{\partial k^2}(t, X_t, k; I) + 2 \frac{\partial^2 BS}{\partial k \partial \sigma}(t, X_t, k; I) \frac{\partial I}{\partial k} \\
&+ \frac{\partial^2 BS}{\partial \sigma^2}(t, X_t, k; I) \left(\frac{\partial I}{\partial k}\right)^2 + \frac{\partial BS}{\partial \sigma}(t, X_t, k; I) \frac{\partial^2 I}{\partial k^2}.
\end{aligned}$$

By Proposition 1, the last equality becomes

$$\frac{\partial BS}{\partial \sigma}(t, X_t, k_t^*, I(t, x_t^*)) \frac{\partial^2 I}{\partial k^2}(t, k_t^*) = \frac{\partial^2}{\partial k^2} V_t|_{k = k_t^*} - \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I(t, x_t^*)).$$

Thus (2) gives

$$\begin{aligned}
&\frac{\partial BS}{\partial \sigma}(t, X_t, k_t^*, I(t, k_t^*)) \frac{\partial^2 I}{\partial k^2}(t, k_t^*) \\
&= E_t \left[\frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, v_t)\right] - \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I(t, k_t^*)). \qquad (4)
\end{aligned}$$

But the last term on the right-hand side of (4) can be written as

$$\begin{aligned}
&\frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, I(t, k_t^*)) \\
&= \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}(V_t)) \\
&= \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}(E_t(BS(t, X_t, k_t^*, v_t)))),
\end{aligned}$$

where, in this case, we denote $BS^{-1}(t, x_t^*, k, \cdot)$ by $BS^{-1}(\cdot)$ in order to simplify the notation. Consequently, using (4), we can establish

$$\begin{aligned}
&\frac{\partial BS}{\partial \sigma}\left(t, X_t, k_t^*, I(t, k_t^*)\right) \frac{\partial^2 I}{\partial k^2}\left(t, k_t^*\right) \\
&= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, v_t\right) - \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right))\right] \\
&= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right. \\
&\quad \left. - \frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right))\right]. \qquad (5)
\end{aligned}$$

Now the proof is decomposed into several steps.

Step 1. Let us first prove that $\frac{\partial^2 I}{\partial k^2}\left(t, k_t^*\right)$ is positive.

The Clark-Ocone formula (see Nualart (2006)), together with Hypotheses (H1) and (H3), leads to

$$BS\left(t, X_t, k_t^*, v_t\right) = E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right) + \int_t^T U_r dB_r,$$

where

$$\begin{aligned}
U_r &: = E_r\left(D_r^B\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right) \\
&= E_r\left(\left(\frac{\partial BS}{\partial \sigma}\left(t, X_t, k_t^*, v_t\right)\right) \frac{D_r^B M_T}{2(T-t)v_t}\right), \quad r > t. \qquad (6)
\end{aligned}$$

Hence, utilizing the convention $\Psi\left(a\right) := \frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(a\right)\right)$ and equality

(5), we get

$$\begin{aligned}
&\frac{\partial BS}{\partial \sigma}\left(t, X_t, k_t^*, I(t,k_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t,k_t^*\right)\\
&= E_t\left[\frac{\partial^2 BS}{\partial k^2}\left(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)+\int_t^T U_r dB_r\right)\right)\right.\\
&\quad \left.-\frac{\partial^2 BS}{\partial k^2}(t, X_t, k_t^*, BS^{-1}\left(E_t\left(BS\left(t, X_t, k, v_t\right)\right)\right))\right]\\
&= E_t\left[\int_t^T \frac{\partial \Psi}{\partial a}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)+\int_t^u U_r dB_r\right)U_u dB_u\right.\\
&\quad \left.+\frac{1}{2}\int_t^T \frac{\partial^2 \Psi}{\partial a^2}\left(E_t\left(BS\left(t, X_t, k_t^*, v_t\right)\right)+\int_t^u U_r dB_r\right)U_u^2 du\right]\\
&= E_t\left[\int_t^T \frac{\partial \Psi}{\partial a}\left(E_u\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)U_u dB_u\right.\\
&\quad \left.+\frac{1}{2}\int_t^T \frac{\partial^2 \Psi}{\partial a^2}\left(E_u\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)U_u^2 du\right]\\
&= \frac{1}{2}E_t\left[\int_t^T \frac{\partial^2 \Psi}{\partial a^2}\left(E_u\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)U_u^2 du\right],
\end{aligned}$$

where, in the last equality, we use the fact that $\frac{\partial \Psi}{\partial a}(a) = \frac{1}{4} - \frac{1}{a^2(T-t)}$, and hypotheses (H1) and (H3). So, now it is easy to see

$$\begin{aligned}
&\frac{\partial BS}{\partial \sigma}\left(t, x_t^*, k, I(t,x_t^*)\right)\frac{\partial^2 I}{\partial k^2}\left(t,x_t^*\right)\\
&= E_t\left[e^{-X_t}\int_t^T \frac{\sqrt{2\pi}\exp\left(\frac{\left(BS^{-1}\left(E_u\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right)^2 (T-s)}{8}\right)}{\left(BS^{-1}\left(E_u\left(BS\left(t, X_t, k_t^*, v_t\right)\right)\right)\right)^3 (T-t)^{3/2}}U_u^2 du\right].
\end{aligned}$$

Thus $\frac{\partial^2 I}{\partial k^2}\left(t, x_t^*\right) > 0$ w.p.1.

Step 2. Here we show that

$$\frac{E_t\left[\int_t^T \frac{\partial^2 \Psi}{\partial a^2}(t,\Lambda_r)\left(U_r^2 - \left(E_r\left(\partial_\sigma BS(t, X_t, k_t^*, v_t)\frac{(T-r)D^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2\right)dr\right]}{(T-t)^{1/2}},$$

where Λ_r defined as in Lemma 4, converges to 0 as $t \uparrow T$.

By Schwarz inequality, we can write

$$
\begin{aligned}
&\left| U_r^2 - \left(E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{(T-r) D^+ \sigma_t^2}{2(T-t) v_t} \right) \right)^2 \right| \\
&= \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t) v_t} \int_r^T (D_r^B \sigma_u^2 + D_t^+ \sigma_t^2) du \right) \right| \\
&\quad \times \left| E_r \left(\partial_\sigma BS(t, X_t, k_t^*, v_t) \frac{1}{2(T-t) v_t} \int_r^T (D_r^B \sigma_u^2 - D_t^+ \sigma_t^2) du \right) \right| \\
&\leq \frac{C e^{2X_t}}{(T-t)} \left| E_r \left(\frac{\exp\left(-\frac{v_t^2 (T-t)}{8} \right)}{v_t} \int_r^T (D_r^B \sigma_u^2 + D_t^+ \sigma_t^2) du \right) \right| \\
&\quad \times \left| E_r \left(\frac{\exp\left(-\frac{v_t^2 (T-t)}{8} \right)}{v_t} \int_r^T (D_r^B \sigma_u^2 - D_t^+ \sigma_t^2) du \right) \right| \\
&\leq \frac{C e^{2X_t}}{(T-t)} E_r \left(\frac{\exp\left(-\frac{v_t^2 (T-t)}{4} \right)}{v_t^2} \right) \left(E_r \left(\left[\int_r^T (D_r^B \sigma_u^2 + D_t^+ \sigma_t^2) du \right]^2 \right) \right)^{1/2} \\
&\quad \times \left(E_r \left(\left[\int_r^T (D_r^B \sigma_u^2 - D_t^+ \sigma_t^2) du \right]^2 \right) \right)^{1/2} \\
&\leq \frac{C (T-r)^2 e^{2X_t}}{(T-t)} E_r \left(\frac{\exp\left(-\frac{v_t^2 (T-t)}{4} \right)}{v_t^2} \right) \left(\sup_{r \leq u \leq T} E_r \left((D_r^B \sigma_u^2 + D_t^+ \sigma_t^2)^2 \right) \right)^{1/2} \\
&\quad \times \left(\sup_{r \leq u \leq T} E_r \left((D_r^B \sigma_u^2 - D_t^+ \sigma_t^2)^2 \right) \right)^{1/2}.
\end{aligned}
$$

Hence, Lemma 4 gives

$$
\begin{aligned}
&\frac{E_t\left[\int_t^T \frac{\partial^2\Psi}{\partial a^2}(t,\Lambda_r)\left(U_r^2-\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)D_t^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2\right)dr\right]}{(T-t)^{1/2}}\\
&\leq \frac{Ce^{X_t}}{(T-t)}E_t\left[\int_t^T E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_t^2(T-t)}{8}\right)}{v_t^3}\right)E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right.\\
&\qquad\left.\times\left(\sup_{r\leq u\leq T}E_r\left((D_r^B\sigma_u^2+D_t^+\sigma_t^2)^2\right)\right)^{1/2}\left(\sup_{r\leq u\leq T}E_r\left((D_r^B\sigma_u^2-D_t^+\sigma_t^2)^2\right)\right)^{1/2}dr\right]\\
&\leq \frac{Ce^{X_t}}{(T-t)}\left(E_t\int_t^T\left(E_r\left(\exp\left(\frac{v_t^2(T-t)}{8}\right)\right)E_r\left(\frac{\exp\left(\frac{3v_t^2(T-t)}{8}\right)}{v_t^3}\right)\right.\right.\\
&\qquad\left.\left.\times E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right)^2dr\right)^{1/2}\\
&\qquad\times\left(E_t\int_t^T\sup_{r\leq u\leq T}\left|E_r\left((D_r^B\sigma_u^2+D_t^+\sigma_t^2)\right)\right|^4dr\right)^{1/4}\\
&\qquad\times\left(E_t\int_t^T\sup_{r\leq u\leq T}\left|E_r\left((D_r^B\sigma_u^2-D_t^+\sigma_t^2)\right)\right|^4dr\right)^{1/4}\\
&\to\quad 0,\quad \text{as } t\uparrow T,
\end{aligned}
$$

due to Hypotheses (H2) and (H3). Thus the claim of this part of the proof is true.

Step 3. Finally we prove that $\lim_{t\to T}\frac{\partial^2 I}{\partial k^2}(t,x_t^*)=\frac{(D_t^+\sigma_t^2)^2}{12\sigma_t^5}$.

From Step 2, we obtain

$$
\begin{aligned}
&\lim_{t\to T}\frac{\partial^2 I}{\partial k^2}(t,x_t^*)\\
&=\lim_{t\to T}\frac{E_t\left[\int_t^T\frac{\partial^2\Psi}{\partial a^2}(t,\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)D^+\sigma_t^2}{2(T-t)v_t}\right)\right)^2dr\right]}{\frac{1}{\sqrt{2\pi}}(T-t)^{1/2}e^{X_t}}\\
&=\lim_{t\to T}\frac{E_t\left[(D^+\sigma_t^2)^2\int_t^T\frac{\partial^2\Psi}{\partial a^2}(t,\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2dr\right]}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}.
\end{aligned}
$$

Note that the right continuity of σ and (H3) imply

$$\frac{(D^+\sigma_t^2)^2 \int_t^T \frac{\partial^2 \Psi}{\partial a^2}(t,\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2 dr}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}$$
$$\rightarrow \quad \frac{(D^+\sigma_t^2)^2}{12\sigma_t^5} \quad \text{as } t \rightarrow T, \quad \text{w.p.1},$$

and

$$\frac{(D^+\sigma_t^2)^2 \int_t^T \frac{\partial^2 \Psi}{\partial a^2}(t,\Lambda_r)\left(E_r\left(\partial_\sigma BS(t,X_t,k_t^*,v_t)\frac{(T-r)}{2v_t}\right)\right)^2 dr}{\frac{1}{\sqrt{2\pi}}(T-t)^{5/2}e^{X_t}}$$
$$\leq \quad \frac{(D^+\sigma_t^2)^2}{(T-t)}\int_t^T E_r\left(\exp\left(\frac{v_r^2(T-t)}{8}\right)\right) E_r\left(\frac{\exp\left(\frac{3v_r^2(T-t)}{8}\right)}{v_t^3}\right) E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right) dr.$$

Therefore, the result follows from the dominated convergence theorem and the fact that

$$\frac{1}{(T-t)}\int_t^T E_t\left(\left(E_r\left(\exp\left(\frac{v_r^2(T-t)}{8}\right)\right) E_r\left(\frac{\exp\left(\frac{3v_r^2(T-t)}{8}\right)}{v_t^3}\right)\right.\right.$$
$$\left.\left.\times E_r\left(\frac{\exp\left(-\frac{v_t^2(T-t)}{4}\right)}{v_t^2}\right)\right)^2\right) dr \rightarrow E_t\left(\frac{1}{\sigma_t^{10}}\right), \quad \text{as } t \rightarrow T,$$

which follows from (H3). Now the proof is finished. ■

Remark 6 *Notice that (H3) can be substituted by adequate integrability conditions.*

Remark 7 *The above theorems proves that, fixed* $t \in [0,T]$*, the implied volatility* $I(t,k)$ *is a locally convex function of the srike with a minimum at* $k = k_t^*$*, according to the previous results by Renault and Touzi (1996).*

5 Examples

5.1 Diffusion stochastic volatilities

Assume that the squared volatility σ^2 can be written as $\sigma^2 = f(Y)$, where $f \in \mathcal{C}_b^1$ and Y is the solution of a stochastic differential equation:

$$dY_r = a(r,Y_r)\,dr + b(r,Y_r)\,dB_r, \tag{7}$$

for some real funcions $a, b \in \mathcal{C}_b^1$. Then, classical arguments (see for example Nualart, 2006) give us that $Y \in \mathbb{L}_B^{1,2}$ and that

$$D_s^B Y_r = \int_s^r \frac{\partial a}{\partial x}(u, Y_u) D_s^B Y_u du + b(s, Y_s) + \int_s^r \frac{\partial b}{\partial x}(u, Y_u) D_s^B Y_u dB_u. \qquad (8)$$

Taking now into account that $D_s^B \sigma_u = f'(Y_u) D_s^B Y_u$ it can be easily deduced from (8) that (H1), and (H3) hold and that

$$\lim_{t \uparrow T} \frac{\int_t^T E_t \left(\sup_{r \leq u \leq T} \left| E_r \left(D_r^B \sigma_u^2 - f'(Y_t) b(t, Y_t) \right) \right|^4 \right) dr}{T - t} = 0,$$

which implies (H2). Then, Theorem1 and Theorem 2 give us that, for all $t \in [0, T]$, $\frac{\partial I}{\partial k}(t, k_t^*) = 0$ and $\frac{\partial^2 I}{\partial k^2}(t, k_t^*) \geq 0$, which proves that $I(t, k)$ is a locally convex function with a minimun at $k = k_t^*$. Moreover, Theorem 2 say us that

$$\lim_{t \to T} \frac{\partial^2 I}{\partial k^2}(t, k_t^*) = \frac{(D_t^+ \sigma_t^2)^2}{12 \sigma_t^5} = \frac{\left(f'(Y_t) b(t, Y_t) \right)^2}{12 \sigma_t^5}.$$

5.2 Fractional stochastic volatility models

Assume that the squared volatility σ^2 can be written as $\sigma^2 = f(Y)$, where $f \in \mathcal{C}_b^1$ and Y is a process of the form

$$Y_r = m + (Y_t - m) e^{-\alpha(r-t)} + c\sqrt{2\alpha} \int_t^r \exp\left(-\alpha (r - s)\right) dB_s^H, \qquad (9)$$

where $B_s^H := \int_0^s (s - u)^{H - \frac{1}{2}} dB_u$. As in Comte and Renault (1998), where this class of models have been introduced to capture the long-time behaviour of the implied volatility, we assume the volatility model (9), for some $H > 1/2$. Notice that (see for example Alòs, Mazet and Nualart (2000)) $\int_t^r \exp\left(-\alpha (r - s)\right) dB_s^H$ can be written as

$$\left(H - \frac{1}{2} \right) \int_t^r \left(\int_s^r \exp\left(-\alpha (r - u)\right) (u - s)^{H - \frac{1}{3}} du \right) dB_s,$$

from where it follows easily that $sup_{t \leq s \leq r \leq T} \left| E\left(D_s^B \sigma_r \middle| \mathcal{F}_s \right) \right| \to 0$ as $t \to T$. In a similar way as in Example 5.1, Theorem 1 and Theorem 2 give us that, for all $t \in [0, T]$, $\frac{\partial I}{\partial k}(t, k_t^*) = 0$ and $\frac{\partial^2 I}{\partial k^2}(t, k_t^*) \geq 0$, and then $I(t, k)$ is a locally convex function with a minimun at $k = k_t^*$. Moreover, Theorem 2 say us that $\lim_{t \to T} \frac{\partial^2 I}{\partial k^2}(t, k_t^*) = 0$. This result indicates that, in order to capture both the short-time and the long time behaviour of the implied volatility, a possible approach could be to consider a volatility process of the form $\sigma^2 = f(Y^1, Y^2)$, where Y^1 is a solution of (7) and Y^2 is a solution of (9), as considered in a recent paper by Alòs and Yang (2014).

6 Conclusions

We have seen, by a computation of the corresponding first and second derivatives, that in the case of uncorrelated volatility, the implied volatility is a locally convex function of the strike, with a minimum at the forward price of the stock. This recovers the previous results by Renault and Touzi (1996). Moreover, we see that the short-time limit of the at-the-money second derivative can be explicitly computed in terms of the Malliavin derivative of the volatility process. Our analysis only needs some general integrability and regularity conditions in the Malliavin calculus sense and does not need the volatility to be a diffusion or a Markov process

References

[1] Alòs, E., León, J.A. and Vives, J.: On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility. Finance Stoch. **11**, 571-589, 2007.

[2] Alòs, E, Mazet, O. and Nualart, D: : Stochastic Calculus with respect to Gaussian processes. The Annals of Probability **29** (2), 766-801, 2001.

[3] Alòs, E. and Yang, Y.: A closed-form option pricing approximation formula for a fractional Heston model. Working Paper **1446**, Universitat Pompeu Fabra, 2014.

[4] Comte, F. and Renault, E. : Long-memory in continuous-time stochastic volatility models. *Mathematical Finance* **8**, 291-323, 1998.

[5] Hull, J. C. and White, A.: The pricing of options on assets with stochastic volatilities. Journal of Finance **42**, 281-300, 1987.

[6] Lee, R.: *Implied volatility: statics, dynamics and probabilistic interpretation*, in *Recent advances in applied probability*. Springer, 2005.

[7] Nualart, D.: The Malliavin Calculus and Related Topics. Second Edition. Probability and its Applications. Springer-Verlag, 2006.

[8] Renault, E. and Touzi, N: Option Hedging and Implicit Volatilities, Mathematical Finance **6**, 279-302, 1996.