On the density of systems of non-linear spatially homogeneous SPDEs

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Abstract

In this paper, we consider a system of k second order non-linear stochastic partial differential equations with spatial dimension $d \geq 1$, driven by a q-dimensional Gaussian noise, which is white in time and with some spatially homogeneous covariance. The case of a single equation and a one-dimensional noise, has largely been studied in the literature. The first aim of this paper is to give a survey of some of the existing results. We will start with the existence, uniqueness and Hölder's continuity of the solution. For this, the extension of Walsh's stochastic integral to cover some measure-valued integrands will be recalled. We will then recall the results concerning the existence and smoothness of the density, as well as its strict positivity, which are obtained using techniques of Malliavin calculus. The second aim of this paper is to show how these results extend to our system of SPDEs. In particular, we give sufficient conditions in order to have existence and smoothness of the density on the set where the columns of the diffusion matrix span $\mathbb{R}^k$. We then prove that the density is strictly positive in a point if the connected component of the set where the columns of the diffusion matrix span $\mathbb{R}^k$ which contains this point has a non void intersection with the support of the law of the solution. We will finally check how all these results apply to the case of the stochastic heat equation in any space dimension and the stochastic wave equation in dimension $d \in \{1, 2, 3\}$.

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1 Introduction

Consider the system of stochastic partial differential equations:

$$Lu_i(t,x) = \sum_{j=1}^{q} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)), \quad t \geq 0, \; x \in \mathbb{R}^d, \tag{1.1}$$

$i = 1, ..., k$, with vanishing initial conditions. Here L is a second order differential operator, and $\sigma_{ij}, b_i : \mathbb{R}^k \mapsto \mathbb{R}$ are globally Lipschitz functions, which are the entries of a $k \times q$ matrix σ and a k-dimensional vector b. We denote by $\sigma_1, ..., \sigma_q$ the columns of the matrix σ. The driving perturbation $\dot{W}(t,x) = (\dot{W}^1(t,x), ..., \dot{W}^q(t,x))$ is a q-dimensional Gaussian noise which is white in time and with a spatially homogeneous covariance f, that is,

$$\mathrm{E}\,[\dot{W}^i(t,x)\dot{W}^j(s,y)] = \delta(t-s)f(x-y)\delta_{ij},$$

where $\delta(\cdot)$ denotes the Dirac delta function, δ_{ij} the Kronecker symbol, and f is a positive continuous function on $\mathbb{R}^d \setminus \{0\}$.

The basic examples we are interested in are the stochastic wave and heat equations with vanishing initial conditions, that is, $L = \frac{\partial^2}{\partial t^2} - \Delta$ and $L = \frac{\partial}{\partial t} - \Delta$, where Δ denotes the Laplacian operator in $\mathbb{R}^d$, and the spatial covariance f to be the Riesz kernel, that is, $f(x) = \|x\|^{-\beta}$, $0 < \beta < d$.

Let $(\mathscr{F}_t)_{t \geq 0}$ denote the filtration generated by W, and let $T > 0$ be fixed. By definition, the solution to the formal equation (1.1) is an adapted stochastic process $\{u(t,x) = (u_1(t,x), ..., u_k(t,x)), (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that

$$\begin{aligned} u_i(t,x) = & \sum_{j=1}^{q} \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma_{ij}(u(s,y))W^j(ds,dy) \\ & + \int_0^t \int_{\mathbb{R}^d} b_i(u(t-s, x-y))\Gamma(s,dy)ds, \end{aligned} \tag{1.2}$$

where Γ denotes the fundamental solution of the deterministic equation $Lu = 0$.

Recall that when $\Gamma(t,x)$ is a real valued function, the stochastic integral appearing in (1.2) is the classical Walsh stochastic integral (see [27]). However, when Γ is measure, Dalang [5] extended Walsh's stochastic integral using techniques of Fourier analysis, and covered, for instance, the case of the wave equation in dimension three. D.Nualart and Quer-Sardanyons [19] extend Walsh's stochastic integral using techniques of stochastic integration with respect to a cylindrical Wiener process (see [12]), in order to cover some classes of measure-valued integrands.

In this paper, we are interested in studying the existence, smoothness and strict positivity of the density of the solution to the system of SPDEs (1.1), on the set where $\{\sigma_1, ..., \sigma_q\}$ span $\mathbb{R}^k$. The case of a single equation has largely been studied in the literature, so our aim is to give a survey of the known results and explain how they extend to the case of a system of equations.

One of the motivations of the results of this article, is to develop in a further work potential theory for solutions of systems of the type (1.1) (see [9]). For the moment potential theory for systems of non-linear SPDEs has been studied by Dalang and E.Nualart in [6] for the wave equation, and by Dalang, Khoshnevisan and E.Nualart in [7] and [8], for the heat equation, all driven by space-time white noise. That is, taking in equation (1.1), $d = 1$, f the Dirac delta function and L the fundamental solution of the deterministic wave or heat equation. In all these works, the existence, smoothness, and strict positivity of the density of the solution to the system of SPDEs is required.

The paper is organized as follows. Section 2 deals with the existence and uniqueness of the solution to (1.2). For this, we first need to define rigorously the noise W, and explain in which sense the stochastic integral in (1.2) is understood, recalling the mentioned results in [5] and [19]. Studying the density of an SPDE is sometimes related to the Hölder continuity properties of the paths of its solution. Thus, we will recall some results concerning the Hölder continuity of the solution to the stochastic heat and wave equations. This will be done in Section 3. Section 4 will be devoted to the study of the existence and smoothness of the density of the solution to (1.1). For this, some elements of Malliavin calculus need to be introduced. Finally, the aim of Section 5 is to show the strict positivity of the density. This last result turns out to be new in the literature. Its proof extends the method used by Bally and Pardoux in [1] for the case of the stochastic heat equation driven by a space-time white noise, in our spatially homogeneous situation.

2 The stochastic integral

The aim of this section is to recall the results concerning the existence and uniqueness of the solution to our class of SPDEs (1.1). For this, we will first recall in our q-dimensional situation the extension of Walsh's stochastic integral given in [19], that uses Da Prato and Zabczyk's stochastic integration theory with respect to cylindrical Wiener processes. In the recent paper [11], Dalang and Quer-Sardanyons extensively explain how this extension is related to the pioneer work by Dalang in [5].

Fix a time interval $[0, T]$. The Gaussian random perturbation $W = (W^1, ..., W^q)$ is a q-dimensional zero mean Gaussian family of random variables $W = \{W^j(\varphi), 1 \leq j \leq q, \varphi \in \mathscr{C}_0^\infty([0,T] \times \mathbb{R}^d; \mathbb{R})\}$, defined on a complete probability $(\Omega, \mathscr{F}, \mathrm{P})$ with covariance

$$\mathrm{E}[W^i(\varphi)W^j(\psi)] = \delta_{ij} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y)\, dxdydt,$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a continuous function on $\mathbb{R}^d \backslash \{0\}$ which is integrable in a neighborhood of 0.

Observe that in order that there exists a Gaussian process with this covariance functional, it is necessary and sufficient that the covariance functional is non-negative definite. This implies that f is symmetric, and is equivalent to the fact that it is the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$. That is,

$$f(x) = \mathscr{F}\mu(x) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} \mu(d\xi),$$

and for some integer $m \geq 1$,

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + \|\xi\|^2)^m} < +\infty.$$

Elementary properties of the convolution and Fourier transform show that this covariance can also be written in terms of the measure μ as

$$\mathrm{E}[W^i(\varphi)W^j(\psi)] = \delta_{ij}\int_0^T\int_{\mathbb{R}^d}\mathscr{F}\varphi(t)(\xi)\overline{\mathscr{F}\psi(t)(\xi)}\mu(d\xi)dt,$$

where $\overline{\mathscr{F}\psi}$ denotes the complex conjugate of $\mathscr{F}\psi$.

Let $\mathscr{H}^q$ denote the completion of the Schwartz space $\mathscr{S}(\mathbb{R}^d;\mathbb{R}^q)$ of $\mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R}^q)$ functions with rapid decrease, endowed with the inner product

$$\langle\varphi,\psi\rangle_{\mathscr{H}^q} = \sum_{\ell=1}^q\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi_\ell(x)f(x-y)\psi_\ell(y)\,dxdy = \sum_{\ell=1}^q\int_{\mathbb{R}^d}\mathscr{F}\varphi_\ell(\xi)\overline{\mathscr{F}\psi_\ell(\xi)}\mu(d\xi),$$

$\phi,\psi\in\mathscr{S}(\mathbb{R}^d;\mathbb{R}^q)$. Notice that $\mathscr{H}^q$ may contain distributions. Set $\mathscr{H}_T^q = L^2([0,T];\mathscr{H}^q)$. In particular, $\mathscr{H}_T^q$ is the completion of the space $\mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^q)$ with respect to the scalar product

$$\langle\varphi,\psi\rangle_{\mathscr{H}_T^q} = \sum_{\ell=1}^q\int_0^T\int_{\mathbb{R}^d}\mathscr{F}\varphi_\ell(t)(\xi)\overline{\mathscr{F}\psi_\ell(t)(\xi)}\mu(d\xi)dt.$$

Then the Gaussian family W can be extended to $\mathscr{H}_T^q$ and we will use the same notation $\{W(g), g\in\mathscr{H}_T^q\}$, where $W(g) = \sum_{i=1}^q W^i(g_i)$.

Now, set $W_t(h) = \sum_{i=1}^q W^i(1_{[0,t]}h_i)$, for any $t\geq 0$, $h\in\mathscr{H}^q$. Then $\{W_t, t\in[0,T]\}$ is a cylindrical Wiener process in the Hilbert space $\mathscr{H}^q$ (cf. [12, Section 4.3.1]). That is, for any $h\in\mathscr{H}^q$, $\{W_t(h), t\in[0,T]\}$ is a Brownian motion with variance $t\|h\|^2_{\mathscr{H}^q}$, and

$$\mathrm{E}[W_s(h)W_t(g)] = (s\wedge t)\langle h,g\rangle_{\mathscr{H}^q}.$$

Let $(\mathscr{F}_t)_{t\geq 0}$ denote the σ-field generated by the random variables $\{W_s(h), h\in\mathscr{H}^q, 0\leq s\leq t\}$ and the P-null sets. We define the predictable σ-field as the σ-field in $\Omega\times[0,T]$ generated by the sets $\{(s,t]\times A, 0\leq s<t\leq T, A\in\mathscr{F}_s\}$. Then following [12, Chapter 4], we can define the stochastic integral of any predictable process $g\in L^2(\Omega\times[0,T];\mathscr{H}^q)$ with respect to the cylindrical Wiener process W as

$$\int_0^T\int_{\mathbb{R}^d} g\cdot dW := \sum_{j=1}^\infty\int_0^T\langle g_t, e_j\rangle_{\mathscr{H}^q}dW_t(e_j),$$

where (e_j) is an orthonormal basis of $\mathscr{H}^q$. Moreover, the following isometry property holds:

$$\mathrm{E}\Big[\Big|\int_0^T\int_{\mathbb{R}^d} g\cdot dW\Big|^2\Big] = \mathrm{E}\Big[\int_0^T\|g_t\|^2_{\mathscr{H}^q}dt\Big].$$

We next introduce the following condition on the fundamental solution of $Lu=0$, Γ.

(H1) For all $t>0$, $\Gamma(t)$ is a nonnegative distribution with rapid decrease, such that

$$\int_0^T\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t)(\xi)|^2\mu(d\xi)dt < +\infty. \tag{2.1}$$

Moreover, Γ is a nonnegative measure of the form $\Gamma(t, dx)dt$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \Gamma(t, dx) \leq C_T < +\infty. \tag{2.2}$$

Then [19, Lemma 3.2 and Proposition 3.3.] show that under condition **(H1)**, Γ belongs to $\mathscr{H}_T$, and one can define the stochastic integral of a predictable process $G = G(t, dx) = Z(t, x)\Gamma(t, dx) \in L^2(\Omega \times [0, T]; \mathscr{H}^q)$ with respect to W, provided that $Z = \{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ is a predictable process such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[\|Z(t, x)\|^2] < +\infty.$$

In this case, we denote the stochastic integral as

$$\int_0^T \int_{\mathbb{R}^d} G(s, y) \cdot W(ds, dy) = \int_0^T \int_{\mathbb{R}^d} \Gamma(s, y)Z(s, y) \cdot W(ds, dy).$$

We next state the existence and uniqueness result of the solution to equation (1.2). This is an extension to system of SPDEs of [19, Theorem 4.1] (see also [5, Theorem 13]) and can be proved in the same way.

Theorem 2.1. *Under condition* **(H1)**, *there exists a unique adapted process* $\{u(t, x) = (u_1(t, x), ..., u_k(t, x)), (t, x) \in [0, T] \times \mathbb{R}^d\}$ *solution of equation* (1.2), *which is continuous in* L^2 *and satisfies that for all* $T > 0$ *and* $p \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[\|u(t, x)\|^p] < +\infty. \tag{2.3}$$

The basic examples we are interested in are the stochastic heat and wave equations. More precisely, it is well-known (see for e.g. [5]) that if L is the heat operator in $\mathbb{R}^d$, $d \geq 1$, that is, $L = \frac{\partial}{\partial t} - \Delta$, where Δ denotes the Laplacian operator in $\mathbb{R}^d$, or if L is the wave operator in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, that is, $L = \frac{\partial^2}{\partial t^2} - \Delta$, condition **(H1)** is satisfied if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + \|\xi\|^2)} < +\infty. \tag{2.4}$$

For instance, one can take f to be a Riesz kernel, that is, $f(x) = \|x\|^{-\beta}$, $0 < \beta < d$. Then, $\mu(d\xi) = c_{d,\beta}\|\xi\|^{\beta-d}d\xi$, and (2.4) holds if and only if $0 < \beta < (2 \wedge d)$.

3 Hölder continuity of the solution

Studying existence, smoothness and strict positivity of the density of the solution to the system of equations (1.1), will require to know the behaviour of the moments of the increments of the solution, which in particular implies the Hölder continuity of the paths. Let us introduce the following conditions.

(H2) There exists $\gamma_1 > 0$ such that for all $t \in [0,T]$, $h \in [0, T-t]$, $x \in \mathbb{R}^d$, and $p > 1$,

$$\mathrm{E}\left[\|u(t+h,x) - u(t,x)\|^p\right] \leq c_{p,T} h^{\gamma_1 p},$$

for some constant $c_{p,T} > 0$. In particular, for all $\delta > 0$ the trajectories of u are $(\gamma_1 - \delta)$-Hölder continuous in time.

(H3) There exists $\gamma_2 > 0$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$, and $p > 1$,

$$\mathrm{E}\left[\|u(t,x) - u(t,y)\|^p\right] \leq c_{p,T} \|x-y\|^{\gamma_2 p},$$

for some constant $c_{p,T} > 0$. In particular, for all $\delta > 0$ the trajectories of u are $(\gamma_2 - \delta)$-Hölder continuous in space.

Using Kolmogorov's continuity theorem, Sanz-Solé and Sarrà prove in [24] that if there exists $\epsilon \in (0,1)$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + \|\xi\|^2)^\epsilon} < +\infty, \tag{3.1}$$

then the stochastic heat equation satisfies **(H2)** for all $\gamma_1 \in (0, \frac{1-\epsilon}{2})$, and **(H3)** for all $\gamma_2 \in (0, 1-\epsilon)$. In particular, if f is the Riesz kernel, that is $\mu(d\xi) = c_{d,\beta}\|\xi\|^{\beta-d}d\xi$, $0 < \beta < (2 \wedge d)$, then **(H2)** holds for all $\gamma_1 \in (0, \frac{2-\beta}{4})$, and **(H3)** holds for all $\gamma_2 \in (0, \frac{2-\beta}{2})$. Observe that in this case (3.1) holds for all $\epsilon > \frac{\beta}{2}$.

For the stochastic wave equation in dimensions $d \in \{1,2\}$, one obtains that under condition (3.1), **(H2)** and **(H3)** are satisfied for all $\gamma_1, \gamma_2 \in (0, 1-\epsilon)$, that is, $\gamma_1, \gamma_2 \in (0, \frac{2-\beta}{2})$ for the Riesz kernel case. The two-dimensional case was studied by Millet and Sanz-Solé in [16, Proposition 1.4]. See [27] for the one dimensional case.

A different approach based on Sobolev embedding theorem is needed to handle the stochastic wave equation in dimension three. This was done by Dalang and Sanz-Solé in [10] for the case where the spatially homogeneous covariance is defined as a product of a Riesz kernel and a smooth function. In particular, when it is a Riesz kernel, they show that **(H2)** and **(H3)** are satisfied with $\gamma_1, \gamma_2 \in (0, \frac{2-\beta}{2})$.

4 Existence and smoothness of the density

We are now interested in proving existence and smoothness of the density of the solution to the system (1.1) on the set where $\{\sigma_1, ..., \sigma_q\}$ span $\mathbb{R}^k$. As is well known, the Malliavin calculus provides a powerful tool in order to show this kind of results for solutions to SDEs and SPDEs. Let us first recall some existing results for single equations of the type (1.1). For the existence and regularity of the density of the wave equation with $d = 1$ one refers to the work by Carmona and D.Nualart in [3]. The wave equation in spatial dimension 2 was studied by Millet and Sanz-Solé in [16]. The three dimensional case was then handled by Quer-Sardanyons and Sanz-Solé, see [25] and [26]. The case of the stochastic heat equation in any space dimension can be found in [14]. All these results where unified and generalized by D.Nualart and Quer-Sardanyons in [19]. They assumed the following condition.

(H4) There exists $\eta > 0$ such that for all $\tau \in [0,1]$,

$$c\tau^{\eta} \leq \int_0^{\tau} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt,$$

for some constant $c > 0$.

Then they show that if σ, b are $\mathscr{C}^{\infty}$ functions with bounded derivatives of order greater than or equal to one, and $|\sigma| \geq c > 0$, under conditions **(H1)** and **(H4)**, for all $(t,x) \in]0,T] \times \mathbb{R}^d$, the law of $u(t,x)$ has a $\mathscr{C}^{\infty}$ density.

Observe that condition **(H4)** is satisfied for the stochastic heat equation for any $\eta \geq 1$ (see [19, Remark 6.3]), and with $\eta = 3$ for the stochastic wave equation in dimensions $1, 2, 3$ (see [25, (A.3)]).

The proof of this result uses tools of Malliavin calculus. One needs to check first that the random variable $u(t,x)$ is smooth in the Malliavin sense. This can be easily generalized to systems of equations (see Proposition 4.3 below). Secondly, one needs to show that the Malliavin matrix has inverse moments of all orders. For this, one uses condition **(H4)** and the non degeneracy condition on σ. The extension of this result to systems of equations is more delicate as ones deals with a matrix σ instead of a function, and because of that reason we need to consider the following additional assumptions on Γ.

(H5) Conditions **(Hi)**, $i = 1, 2, 3$ are satisfied. Moreover, the nonnegative measure Ψ defined as $\|x\|^{\gamma_2}\Gamma(t,dx)$ satisfies that

$$\int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Psi(t)(\xi)|^2 \mu(d\xi) dt < +\infty.$$

Moreover, there exist $\alpha_1, \alpha_2 > 0$, $\gamma_2 < \alpha_1$, $\gamma_1 < \alpha_2$, such that for all $\tau \in [0,1]$,

$$\int_0^{\tau} \langle \Psi(r,*), \Gamma(r,*) \rangle_{\mathscr{H}} dr \leq c\tau^{\alpha_1},$$

for some constant $c > 0$, and

$$\int_0^{\tau} r^{\gamma_1} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi) dr \leq c\tau^{\alpha_2},$$

for some constant $c > 0$.

(H6) Conditions **(H4)** and **(H5)** are satisfied, and $\alpha := \alpha_1 \wedge \alpha_2 > \eta$.

We will then show the following result.

Theorem 4.1. *Assume hypothesis* **(H6)** *and that σ, b are $\mathscr{C}^{\infty}$ functions with bounded partial derivatives of order greater than or equal to one. Then for all $(t,x) \in]0,T] \times \mathbb{R}^d$, the law of the random vector $u(t,x)$ admits a $\mathscr{C}^{\infty}$ density on the open subset of $\mathbb{R}^k$ $\Sigma := \{y \in \mathbb{R}^k : \sigma_1(y), ..., \sigma_q(y) \text{ span } \mathbb{R}^k\}$. That is, there exists a function $p_{t,x} \in \mathscr{C}^{\infty}(\Sigma; \mathbb{R})$ such that for every bounded and continuous function $f : \mathbb{R}^k \mapsto \mathbb{R}$ with support contained in Σ,*

$$\mathrm{E}[f(u(t,x))] = \int_{\mathbb{R}^k} f(y)\, p_{t,x}(y) dy.$$

We next prove an intermediary result that will be needed for the proof of Theorem 4.1.

Lemma 4.2. *Assume hypothesis* **(H5)**. *Then for all* $t \in [0,T]$, $x \in \mathbb{R}^d$, $\epsilon \in (0,1]$, *and* $p > 1$,

$$\mathrm{E}\left[\left|\int_0^\epsilon \Big\langle (\sigma_{ij}(u(t-r,*)) - \sigma_{ij}(u(t,x)))\Gamma(r,x-*), \Gamma(r,x-*)\Big\rangle_{\mathscr{H}} dr\right|^p\right] \le c_{p,T}\epsilon^{\alpha p},$$

for some constant $c_{p,T} > 0$, *where* $\alpha := \alpha_1 \wedge \alpha_2$ *and* α_1, α_2 *are the parameters in hypothesis* **(H5)**.

Proof. Set $G(r,dy) := (\sigma_{ij}(u(t-r,y)) - \sigma_{ij}(u(t,x)))\Gamma(r,x-dy)$. Then by [19, Proposition 3.3], $G(r,dy) \in L^2(\Omega \times [0,T]; \mathscr{H})$, and following the proof of this Proposition, we can write

$$\langle G(r,x-*), \Gamma(r,x-*)\rangle_{\mathscr{H}} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(r,x-dy)f(y-z)\Gamma(r,x-dz).$$

Using Minkowski's inequality with respect to the finite measure $\Gamma(r,x-dy)\Gamma(r,x-dz)f(y-z)dr$, together with the Lipschitz property of σ, we get that for all $p > 1$,

$$\begin{aligned}&\mathrm{E}\left[\left|\int_0^\epsilon \langle G(r,x-*), \Gamma(r,x-*)\rangle_{\mathscr{H}} dr\right|^p\right] \\ &\le c_p \left|\int_0^\epsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,x-dy)(\mathrm{E}\left[\|u(t-r,y) - u(t,x)\|^p\right])^{1/p}\Gamma(r,x-dz)f(y-z)dr\right|^p.\end{aligned}$$

Therefore, appealing to hypotheses **(H2)** and **(H3)**, we find that the last term is bounded by

$$\begin{aligned}&c_{p,T}\Bigg\{\left|\int_0^\epsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,x-dy)\|x-y\|^{\gamma_2}\Gamma(r,x-dz)f(y-z)dr\right|^p \\ &+ \left|\int_0^\epsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,x-dy)r^{\gamma_1}\Gamma(r,x-dz)f(y-z)dr\right|^p\Bigg\} \\ &= c_{p,T}\Bigg\{\left|\int_0^\epsilon \langle \Psi(r,*), \Gamma(r,*)\rangle_{\mathscr{H}} dr\right|^p + \left|\int_0^\epsilon r^{\gamma_1}\int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi)dr\right|^p\Bigg\}.\end{aligned}$$

Hence, using hypothesis **(H5)**, we conclude that

$$\mathrm{E}\left[\left|\int_0^\epsilon \langle G(r,x-*), \Gamma(r,x-*)\rangle_{\mathscr{H}} dr\right|^p\right] \le c_{p,T}(\epsilon^{\alpha_1 p} + \epsilon^{\alpha_2 p}) \le c_{p,T}\epsilon^{\alpha p}.$$

□

Next, we recall some elements of Malliavin calculus. Consider the Gaussian family $\{W(h), h \in \mathscr{H}_T^q\}$ defined in Section 2, that is, a centered Gaussian process such that $\mathrm{E}[W(h)W(g)] = \langle h, g\rangle_{\mathscr{H}_T^q}$. Then, we can use the differential Malliavin calculus based on it (see for instance [18]). We denote the Malliavin derivative by $D = (D^{(1)}, ..., D^{(q)})$, which is an operator in $L^2(\Omega; \mathscr{H}_T^q)$. For any $m \ge 1$, the domain of the iterated derivative D^m in $L^p(\Omega; (\mathscr{H}_T^q)^{\otimes m})$ is denoted by $\mathbb{D}^{m,p}$, for any $p \ge 2$. We set $\mathbb{D}^\infty = \cap_{p\ge1}\cap_{m\ge1}\mathbb{D}^{m,p}$. Recall that for any differentiable random variable F and any $r = (r_1, ..., r_m) \in [0, T]^m$, $D^m F(r)$ is an element of $\mathscr{H}^{\otimes m}$, which will be denoted by $D^m_r F$. We define the Malliavin matrix of $F \in (\mathbb{D}^\infty)^m$ by $\gamma_F = (\langle DF_i, DF_j \rangle_{\mathscr{H}_T^q})_{1\le i,j\le m}$.

The next result is the q-dimensional extension of [19, Proposition 6.1] (see also [26, Theorem 1]). Its proof follows exactly along the same lines working coordinate by coordinate, and is therefore omitted.

Proposition 4.3. *Assume that* **(H1)** *holds, and that σ, b are $\mathscr{C}^\infty$ functions with bounded partial derivatives of order greater than or equal to one. Then, for every $(t, x) \in (0, T] \times \mathbb{R}^d$, the random variable $u_i(t, x)$ belongs to the space $\mathbb{D}^\infty$, for all $i = 1, ..., k$. Moreover, the derivative $Du_i(t, x) = (D^{(1)}u_i(t, x), ..., D^{(q)}u_i(t, x))$ is an $\mathscr{H}_T^q$-valued process that satisfies the following linear stochastic differential equation, for all $i = 1, ..., k$:*

$$\begin{aligned} D_r^{(j)}u_i(t,x) &= \sigma_{ij}(u(r,*))\Gamma(t-r,x-*) \\ &\quad + \int_r^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) \sum_{\ell=1}^q D_r^{(j)}(\sigma_{i\ell}(u(s,y))W^\ell(ds,dy) \\ &\quad + \int_r^t \int_{\mathbb{R}^d} \Gamma(t-s,dy) D_r^{(j)}(b_i(u(s,x-y))ds, \end{aligned} \tag{4.1}$$

for all $r \in [0, t]$, and is 0 otherwise. Moreover, for all $p \ge 1$, $m \ge 1$ and $i = 1, ..., k$, it holds that

$$\sup_{(t,x)\in(0,T]\times\mathbb{R}^d} \mathrm{E}\Big[\|D^m u_i(t,x)\|^p_{(\mathscr{H}_T^q)^{\otimes m}}\Big] < +\infty \tag{4.2}$$

Remark 4.4. Recall that the iterated derivative of $u_i(t, x)$ satisfies also the q-dimensional extension of equation [19, (6.27)].

Recall that the stochastic integral on the right hand-side of (4.1) must be understood by means of a Hilbert-valued stochastic integral (see [19, Section 3]) and the Hilbert-valued pathwise integral of (4.1) is defined in [19, Section 5].

In order to prove Theorem 4.1, we will use the following localized variant of Malliavin's absolute continuity theorem.

Theorem 4.5. [1, Theorem 3.1] *Let $\Sigma_m \subset \mathbb{R}^k$, $m \in \mathbb{N}$, $m \ge 1$, be a sequence of open sets such that $\bar{\Sigma}_m \subset \Sigma_{m+1}$ and let $F \in (\mathbb{D}^\infty)^k$ such that for any $q > 1$ and $m \in \mathbb{N}$,*

$$\mathrm{E}\big[(\det \gamma_F)^{-q} \mathbf{1}_{\{F\in\Sigma_m\}}\big] < +\infty. \tag{4.3}$$

Then the law of F admits a $\mathscr{C}^\infty$ density on the set $\Sigma = \cup_m \Sigma_m$.

Proof of Theorem 4.1. For each $m \in \mathbb{N}$, $m \ge 1$, define the open set

$$\Sigma_m = \Big\{ y \in \mathbb{R}^k : \sum_{j=1}^q \langle \sigma_j(y), \xi\rangle^2 > \frac{1}{m},\ \forall\, \xi \in \mathbb{R}^k,\ \|\xi\| = 1 \Big\}. \tag{4.4}$$

By Proposition 4.3, it suffices to prove that condition (4.3) of Theorem 4.5 holds true in each Σ_m. Indeed, observe that $\Sigma = \cup_m \Sigma_m = \{\sigma_1, ..., \sigma_q \text{ span } \mathbb{R}^k\}$.

Let $(t,x)\in]0,T]\times\mathbb{R}^d$ and $m\geq 1$ be fixed. It suffices to prove that there exists $\delta_0(m)>0$ such that for all $0<\delta\leq\delta_0$, and all $p>1$,

$$\mathrm{P}\left\{(\det\gamma_{u(t,x)}<\delta)\mathbf{1}_{\{u(t,x)\in\Sigma_m\}}\right\}\leq C\delta^{\lambda p}, \tag{4.5}$$

for some $\lambda>0$, and for some constant $C>0$ not depending on δ. This implies (4.3), taking $p=\frac{q}{\lambda}+1$ in (4.5).

We write

$$\det\gamma_{u(t,x)}\geq\left(\inf_{\xi\in\mathbb{R}^k:\|\xi\|=1}(\xi^T\gamma_{u(t,x)}\xi)\right)^k. \tag{4.6}$$

Let $\xi\in\mathbb{R}^k$ with $\|\xi\|=1$, and fix $\epsilon\in(0,1]$. The inequality

$$\|a+b\|^2_{\mathscr{H}^q}\geq\frac{2}{3}\|a\|^2_{\mathscr{H}^q}-2\|b\|^2_{\mathscr{H}^q},$$

together with (4.1), gives

$$\xi^T\gamma_{u(t,x)}\xi\geq\int_{t-\epsilon}^t\left\|\sum_{i=1}^k D_r(u_i(t,x))\xi_i\right\|^2_{\mathscr{H}^q}dr\geq\frac{2}{3}\mathscr{A}_1-2\mathscr{A}_2,$$

where

$$\begin{aligned}
\mathscr{A}_1&=\sum_{j=1}^q\int_{t-\epsilon}^t\|\langle\sigma_j(u(r,*)),\xi\rangle\Gamma(t-r,x-*)\|^2_{\mathscr{H}^q}dr,\\
\mathscr{A}_2&=\int_{t-\epsilon}^t\left\|\sum_{i=1}^k a_i(r,t,x,*)\,\xi_i\right\|^2_{\mathscr{H}^q}dr,\\
a_i(r,t,x,*)&=\sum_{\ell=1}^q\int_r^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)D_r(\sigma_{i\ell}(u(s,y))W^\ell(ds,dy)\\
&\quad+\int_r^t\int_{\mathbb{R}^d}D_r(b_i(u(s,x-y))\Gamma(t-s,y)dyds.
\end{aligned}$$

Now, assume that $u(t,x)\in\Sigma_m$. Then, adding and subtracting the term

$$\mathscr{A}_{1,1}=\sum_{j=1}^q\langle\sigma_j(u(t,x)),\xi\rangle^2\int_{t-\epsilon}^t\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t-r)(\xi)|^2\mu(d\xi)dr$$

we get that $\mathscr{A}_1\geq\mathscr{A}_{1,1}-|\mathscr{A}_{1,2}|$, where

$$\begin{aligned}
\mathscr{A}_{1,2}=\sum_{j=1}^q\int_{t-\epsilon}^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}&\Gamma(t-r,x-dy)\Gamma(t-r,x-dz)f(y-z)\\
&\times\Big(\langle\sigma_j(u(r,y)),\xi\rangle\langle\sigma_j(u(r,z)),\xi\rangle-\langle\sigma_j(u(t,x)),\xi\rangle^2\Big)\,dr.
\end{aligned}$$

Note that we have added and subtracted a "local" term to make the ellipticity property appear (see (4.4)). A similar idea is used in [16] for the stochastic wave equation in dimension 2.

Then, using the fact that $u(t,x) \in \Sigma_m$, we get that

$$\mathscr{A}_{1,1} > \frac{1}{m} \int_0^\epsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi) dr =: \frac{1}{m} g(\epsilon).$$

Now we find out upper bounds for the p-th moment, $p > 1$, of the terms $\mathscr{A}_{1,2}$ and $\mathscr{A}_2$. We start treating $\mathscr{A}_{1,2}$. We write

$$\begin{aligned}
&\langle \sigma_j(u(r,y)), \xi \rangle \langle \sigma_j(u(r,z)), \xi \rangle - \langle \sigma_j(u(t,x)), \xi \rangle^2 \\
&= \langle \sigma_j(u(r,y)), \xi \rangle \, \langle \sigma_j(u(r,z)) - \sigma_j(u(t,x)), \xi \rangle \\
&\qquad + \langle \sigma_j(u(t,x)), \xi \rangle \, \langle \sigma_j(u(r,y)) - \sigma_j(u(t,x)), \xi \rangle.
\end{aligned}$$

Then, proceeding as in the proof of Lemma 4.2, using the Lipschitz property of the coefficients of σ, and (2.3), it yields that

$$\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\|=1} |\mathscr{A}_{1,2}|^p \right] \le c_{p,T} \epsilon^{\alpha p}.$$

We next treat $\mathscr{A}_2$. Using the Cauchy-Schwarz inequality, for any $p > 1$, it yields that

$$\begin{aligned}
\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\|=1} |\mathscr{A}_2|^p \right] &\le c_p \sum_{i=1}^k \mathrm{E}\left[\left| \int_{t-\epsilon}^t \|a_i(r,t,x,*)\|^2_{\mathscr{H}^q} dr \right|^p \right] \\
&\le c_p \sum_{i=1}^k \left(\mathrm{E}\left[\left| \int_0^\epsilon \|V_i(r,t,x,*)\|^2_{\mathscr{H}^q} dr \right|^p \right] + \mathrm{E}\left[\left| \int_0^\epsilon \|W_i(r,t,x,*)\|^2_{\mathscr{H}^q} dr \right|^p \right] \right),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
V_i(r,t,x,*) &:= \sum_{\ell=1}^q \int_{t-r}^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) D_{t-r}(\sigma_{i\ell}(u(s,y))) W^\ell(ds,dy), \\
W_i(r,t,x,*) &:= \int_{t-r}^t \int_{\mathbb{R}^d} \Gamma(t-s, y) D_{t-r}(b_i(u(s,x-y))) dy ds.
\end{aligned}$$

Now, using Hölder's inequality, the boundedness of the coefficients of the derivatives of σ, and [19, (3.13), (5.26)], we get that

$$\begin{aligned}
\mathrm{E}\left[\left| \int_0^\epsilon \|V_i(r,t,x,*)\|^2_{\mathscr{H}^q} dr \right|^p \right] &\le c_p \epsilon^{p-1} \sup_{(s,y) \in (0,\epsilon) \times \mathbb{R}^d} \mathrm{E}[\|D_{t-\cdot} u_i(t-s,y)\|^{2p}_{\mathscr{H}^q_\epsilon}] \\
&\qquad \times \left(\int_0^\epsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds \right)^p \\
&\le c_p g(\epsilon)^{2p}.
\end{aligned}$$

Moreover, using Hölder's inequality, the boundedness of the partial derivatives of the coefficients of b, hypothesis (2.2), and [19, (5.17), (5.26)], for the second term in (4.7) corresponding to the Hilbert-valued pathwise integral, we obtain that

$$\begin{aligned}\mathrm{E}\left[\left|\int_0^\epsilon \|W_i(r,t,x,*)\|^2_{\mathscr{H}^q}dr\right|^p\right] &\le c_p\epsilon^{p-1}\sup_{(s,y)\in(0,\epsilon)\times\mathbb{R}^d}\mathrm{E}[\|D_{t-\cdot}u_i(t-s,y)\|^{2p}_{\mathscr{H}^q_\epsilon}]\\ &\qquad\times\left(\int_0^\epsilon\int_{\mathbb{R}^d}\Gamma(s,y)dyds\right)\\ &\le c_p\epsilon^p g(\epsilon)^p.\end{aligned}$$

Hence, we conclude that, for any $p>1$,

$$\mathrm{E}\left[\sup_{\xi\in\mathbb{R}^d:\|\xi\|=1}|\mathscr{A}_2|^p\right]\le c_p(g(\epsilon)^{2p}+\epsilon^p g(\epsilon)^p).$$

Appealing to (4.6), we have proved that, on the set $\{u(t,x)\in\Sigma_m\}$,

$$\det\gamma_{u(t,x)} > \left(\frac{2}{3m}g(\epsilon)-I\right)^k, \tag{4.8}$$

where I is a random variable such that for all $p>1$, $\mathrm{E}[|I|^p]\le(\epsilon^{\alpha p}+g(\epsilon)^{2p}+\epsilon^p g(\epsilon)^p)$.

We now choose $\epsilon=\epsilon(\delta,m)$ in such a way that $\delta^{1/k}=\frac{1}{3m}g(\epsilon)$. By hypothesis **(H4)** this implies that $3m\delta^{1/k}\ge c\epsilon^\eta$, that is, $\epsilon\le C\delta^{\frac{1}{\eta k}}$. Then using (4.8), we conclude that for all $0<\delta\le\delta_0$ and $p>1$,

$$\mathrm{P}\Big\{(\det\gamma_{u(t,x)}<\delta)\mathbf{1}_{\{u(t,x)\in\Sigma_m\}}\Big\}\le\frac{\epsilon^{\alpha p}+g(\epsilon)^{2p}+\epsilon^p g(\epsilon)^p}{(\frac{2}{3m}g(\epsilon)-\delta^{1/k})^p}\le C(m)(\delta^{\frac{p}{k}}+\delta^{\frac{p}{\eta k}}+\delta^{\lambda p}),$$

with $\lambda=\frac{\alpha-\eta}{\eta k}$. Recall that $\eta<\alpha$ from hypothesis **(H6)**. This proves (4.5). □

We end this section with the verification of condition **(H6)** for the stochastic heat and wave equations with a Riesz kernel as spatial homogeneous covariance, that is, $f(x)=\|x\|^{-\beta}$ and $\mu(d\xi)=c_{d,\beta}\|\xi\|^{d-\beta}d\xi$, $0<\beta<(2\wedge d)$.

Consider first the stochastic heat equation in any spatial dimension, that is,

$$\Gamma(t,x)=(4\pi t)^{-d/2}\exp\left(-\frac{\|x\|^2}{4t}\right),\qquad t\ge0, x\in\mathbb{R}^d.$$

For all $t\in[0,T]$, using the change variables $[\tilde{y}=\frac{y}{\sqrt{r}},\ \tilde{z}=\frac{z}{\sqrt{r}}]$, it yields that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|y-z\|^{-\beta}\Gamma(r,y)\Gamma(r,z)\,dydzdr\\ &\qquad=\int_0^t r^{-\beta/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\tilde{y}-\tilde{z}\|^{-\beta}\Gamma(1,\tilde{y})\Gamma(1,\tilde{z})d\tilde{y}d\tilde{z}dr=ct^{\frac{2-\beta}{2}}.\end{aligned}$$

Thus, hypothesis **(H4)** is satisfied for $\eta = \frac{2-\beta}{2}$. Furthermore, hypothesis **(H5)** holds with $\alpha_1 = \frac{2-\beta}{2} + \frac{\gamma_2}{2}$ and $\alpha_2 = \frac{2-\beta}{2} + \gamma_1$. Indeed, using the change of variables $[\tilde{y} = \frac{y}{\sqrt{r}}, \tilde{z} = \frac{z}{\sqrt{r}}]$, for all $\tau \in [0,1]$, we have that

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|z\|^{\gamma_2} \|y-z\|^{-\beta} \Gamma(r,z)\Gamma(r,y) dz dy dr \\ &= \int_0^\tau r^{\frac{\gamma_2}{2} - \frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\tilde{z}\|^{\gamma_2} \|\tilde{y} - \tilde{z}\|^{-\beta} \Gamma(1,\tilde{z})\Gamma(1,\tilde{y}) d\tilde{z} d\tilde{y} dr = c\tau^{\frac{2-\beta}{2} + \frac{\gamma_2}{2}},\end{aligned}$$

and

$$\begin{aligned}&\int_0^\tau r^{\gamma_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|y-z\|^{-\beta} \Gamma(r,z)\Gamma(r,y) dz dy dr \\ &= \int_0^\tau r^{\gamma_1 - \frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\tilde{y} - \tilde{z}\|^{-\beta} \Gamma(1,\tilde{z})\Gamma(1,\tilde{y}) d\tilde{z} d\tilde{y} dr = c\tau^{\frac{2-\beta}{2} + \gamma_1}.\end{aligned}$$

Observe that $\alpha = \alpha_1 \wedge \alpha_2 = \frac{2-\beta}{2} + \frac{\gamma_2 \wedge (2\gamma_1)}{2}$, which is bigger that η, thus proves **(H6)**.

We next treat the case of the stochastic wave equation with spatial dimension $d \in \{1,2,3\}$. Let Γ_d be the fundamental solution of the deterministic wave equation in $\mathbb{R}^d$ with null initial conditions. In this case, changing variables $[\tilde{\xi} = r\xi]$, for all $t \in [0,T]$, we get that

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} |\mathscr{F}\Gamma_d(r)(\xi)|^2 \|\xi\|^{\beta-d} d\xi dr &= \int_0^t \int_{\mathbb{R}^d} \frac{\sin^2(2\pi r\|\xi\|)}{4\pi^2\|\xi\|^2} \|\xi\|^{\beta-d} d\xi dr \\ &= \int_0^t r^{2-\beta} \int_{\mathbb{R}^d} \frac{\sin^2(2\pi\|\tilde{\xi}\|)}{4\pi^2\|\tilde{\xi}\|^2} \|\tilde{\xi}\|^{\beta-d} d\tilde{\xi} dr \\ &= ct^{3-\beta}.\end{aligned}$$

Therefore, hypothesis **(H4)** is satisfied for $\eta = 3-\beta$. Notice also that

$$\int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Psi(r)(\xi)|^2 \|\xi\|^{\beta-d} d\xi dr \leq T^{2\gamma_2} \int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Gamma_d(r)(\xi)|^2 \|\xi\|^{\beta-d} d\xi dr < +\infty.$$

Moreover, changing variables $[\tilde{\xi} = r\xi]$, for all $\tau \in [0,1]$, we have that

$$\begin{aligned}&\int_0^\tau \langle \Psi(r,*), \Gamma_d(r,*)\rangle_{\mathscr{H}} dr = \int_0^\tau \int_{\mathbb{R}^d} \mathscr{F}\Psi(r)(\xi) \frac{\sin(2\pi r\|\xi\|)}{2\pi\|\xi\|} \|\xi\|^{\beta-d} d\xi dr \\ &= \int_0^\tau r^{1-\beta} \int_{\mathbb{R}^d} \mathscr{F}\Psi(r)\left(\frac{\tilde{\xi}}{r}\right) \frac{\sin(2\pi\|\tilde{\xi}\|)}{2\pi\|\tilde{\xi}\|} \|\tilde{\xi}\|^{\beta-d} d\tilde{\xi} dr \\ &= \langle \Psi(1,*), \Gamma_d(1,*)\rangle_{\mathscr{H}} \int_0^\tau r^{2-\beta+\gamma_2} dr = c\tau^{3-\beta+\gamma_2},\end{aligned}$$

and

$$\int_0^\tau r^{\gamma_1} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma_d(r)(\xi)|^2 \|\xi\|^{\beta-d} d\xi dr = c\tau^{3-\beta+\gamma_1}.$$

Thus, hypotheses **(H5)** and **(H6)** hold taking $\alpha = 3-\beta+\gamma_1 \wedge \gamma_2 > 3-\beta = \eta$.

5 Strict positivity of the density

The second aim of this article is to show that under the same condition **(H6)**, the density of the law of the solution to the system (1.1) is strictly positive in a point if the connected component of the set where $\{\sigma_1, ..., \sigma_q\}$ span $\mathbb{R}^k$ that contains this point has a non void intersection with the support of the law of the solution (see Theorem 5.1 below). For this, we will first extend in our situation a criterion of strict positiveness of densities proved by Bally and Pardoux in [1], which uses essentially an inverse function type result (see Lemma 5.6) and Girsanov's theorem.

We will then apply this criterion to our system of SPDEs (1.1). In [1], the authors apply their criterion to the density of the law of the random vector $(u(t, x_1), ..., u(t, x_k))$, $0 \le x_1 \le \cdots \le x_k \le 1$, where u denotes the solution to the non-linear stochastic heat equation driven by a space-time white noise, by using a localizing argument. Hence, their situation is a bit different as ours, as we deal with a system of SPDEs and we evaluate the solution at a single point $(t, x) \in]0, T] \times \mathbb{R}^d$. In a similar context, Chaleyat and Sanz-Solé in [4] study the strict positivity of the density of the random vector $(u(t, x_1), ..., u(t, x_k))$, $0 \le x_1 \le \cdots \le x_k \le 1$, where u is the solution to the stochastic wave equation in two spatial dimensions, that is, take in equation (1.1) $d = 2$, $k, q = 1$, and $L = \frac{\partial^2}{\partial t^2} - \Delta$.

We will also apply the criterion of strict positivity to the case of a single equation, that is the solution to (1.1) with $k, q = 1$. We will obtain that the density is strictly positivity in all $\mathbb{R}$ under the same hypotheses that D.Nualart and Quer-Sardanyons obtained in [19] its existence and smoothness (see Theorem 5.3 below), with the addition condition of σ being bounded. Recall that in the recent papers [20] and [21], the same authors obtain lower and upper bounds of Gaussian type for the density of the solution to single spatially homogeneous SPDEs of the class (1.1) in the case where σ is a constant, which in particular shows the strict positivity of the density in the quasi-linear case. A first step to show bounds for the non-linear case is done in the recent preprint [22], where upper and lower bounds for the density of non-linear stochastic heat equations in any space dimension are established.

There are also other many SPDEs for which the strict positivity of its density is studied. For example, the case of non-linear hyperbolic SPDEs has been studied by Millet and Sanz-Solé in [15], Fournier [13] considers a Poisson driven SPDE, and the Cahn-Hilliard stochastic equation is studied by Cardon-Weber in [2].

We next state the main result of this paper.

Theorem 5.1. *Assume hypothesis* **(H6)** *and that σ, b are $\mathscr{C}^\infty$ functions with bounded partial derivatives of order greater than or equal to one and σ is bounded. Then for all $(t, x) \in]0, T] \times \mathbb{R}^d$, the law of the random vector $u(t, x)$ admits a $\mathscr{C}^\infty$ density on $\Sigma := \{y \in \mathbb{R}^k : \sigma_1(y), ..., \sigma_q(y)$ span $\mathbb{R}^k\}$ such that $p_{t,x}(y) > 0$ if the connected component of Σ which contains y has a non void interesection with the support of $u(t, x)$.*

Remark 5.2. In the case where σ is uniformly elliptic, that is, $\|\sigma(y)\xi\|^2 \ge c > 0$, for all $\xi \in \mathbb{R}^q$ and $y \in \mathbb{R}^k$, under the hypotheses of Theorem 5.1, we get that for all $(t, x) \in]0, T] \times \mathbb{R}^d$, the law of the random vector $u(t, x)$ admits a $\mathscr{C}^\infty$ strictly positive density.

The case of a single SPDE of the type (1.1) is as follows.

Theorem 5.3. *Assume that σ, b are $\mathscr{C}^\infty$ functions with bounded derivatives of order greater than or equal to one, and $0 < c \leq |\sigma| \leq C$. Then, under conditions* **(H1)** *and* **(H4)**, *for all $(t,x) \in]0,T] \times \mathbb{R}^d$, the law of $u(t,x)$ has a $\mathscr{C}^\infty$ strictly positive density function.*

Remark 5.4. Observe that we only require the continuity of the density in order to show the strict positivity (see Theorem 5.7 below). Hence the $\mathscr{C}^\infty$ condition on the coefficients can be weakened in order to show the strict positivity of the density.

5.1 Criterion for the strict positivity of the density

As in Section 4, we will study the strict positivity of the density on the set where the columns of the $k \times q$ matrix σ, $\sigma_1, ..., \sigma_q$ span $\mathbb{R}^k$. If $y \in \mathbb{R}^k$ is in this set, then there exists $l_1(y) =: l_1, ..., l_k(y) =: l_k$ such that $\sigma_{l_1}(y), ..., \sigma_{l_k}(y)$ span $\mathbb{R}^k$. Thus it suffices to make all the calculations using $W^{l_1(y)}, ..., W^{l_k(y)}$ and ignoring the other W^j. In order to simplify the notation we will take $l_i = i$.

Given $T > 0$, a predictable process $g \in \mathscr{H}_T^k$ and $z \in \mathbb{R}^k$, we define the process $\hat{W} = (\hat{W}^1, ..., \hat{W}^k)$ as

$$\hat{W}^j(1_{[0,t]}h_j) = W^j(1_{[0,t]}h_j) + z_j \int_0^t \langle h_j(*), g^j(s,*)\rangle_{\mathscr{H}} ds,$$

for any $h \in \mathscr{H}^k$, $j = 1, ..., k$, $t \in [0,T]$.

We set $\hat{W}_t(h) = \sum_{j=1}^k \hat{W}^j(1_{[0,t]}h_j)$. Then, by [12, Theorem 10.14], $\{\hat{W}_t, t \in [0,T]\}$ is a cylindrical Wiener process in $\mathscr{H}^k$ on the probability space $(\Omega, \mathscr{F}, \hat{\mathrm{P}})$, where

$$\frac{d\hat{\mathrm{P}}}{d\mathrm{P}}(\omega) = J_z(\omega), \;\; \omega \in \Omega,$$

where

$$J_z = \exp\Big(-\sum_{j=1}^k z_j \int_0^T \int_{\mathbb{R}^d} g^j(s,y) W^j(ds,dy) - \frac{1}{2}\sum_{j=1}^k z_j^2 \int_0^T \|g^j(s,*)\|_{\mathscr{H}}^2 \, ds \Big). \tag{5.1}$$

Then, for any predictable process $Z \in L^2(\Omega \times [0,T]; \mathscr{H}^k)$ and $j = 1, ..., k$, it yields that

$$\int_0^T \int_{\mathbb{R}^d} Z_j(s,y)\hat{W}^j(ds,dy) = \int_0^T \int_{\mathbb{R}^d} Z_j(s,y) W^j(ds,dy) + z_j \int_0^T \langle Z_j(s,*), g^j(s,*)\rangle_{\mathscr{H}} ds.$$

For any $(t,x) \in [0,T] \times \mathbb{R}^k$, let $\hat{u}^z(t,x)$ be the solution to equation (1.2) with respect to the cylindrical Wiener process $\hat{W}$, that is, for $i = 1, ..., k$,

$$\begin{aligned} \hat{u}_i^z(t,x) &= \sum_{j=1}^k \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma_{ij}(\hat{u}^z(s,y)) W^j(ds,dy) \\ &+ \sum_{j=1}^k z_j \int_0^t \langle \Gamma(t-s, x-\cdot)\sigma_{ij}(\hat{u}^z(s,*)), g^j(s,*)\rangle_{\mathscr{H}} ds \\ &+ \int_0^t \int_{\mathbb{R}^d} b_i(\hat{u}^z(t-s, x-y))\Gamma(s,dy)ds. \end{aligned} \tag{5.2}$$

Then, the law of u under P coincides with the law of $\hat{u}^z$ under $\hat{P}$, which implies that for all nonnegative continuous and bounded function $f : \mathbb{R}^k \mapsto \mathbb{R}$,

$$\mathrm{E}[f(u(t,x))] = \mathrm{E}[f(\hat{u}^z(t,x))J_z]. \tag{5.3}$$

Given a sequence $\{g_n\}_{n\geq 1}$ of predictable processes in $\mathscr{H}_T^k$ and $z \in \mathbb{R}^k$, let $\hat{u}_n^z(t,x)$ be the solution to equation (1.2) with respect to the cylindrical Wiener process $\{\hat{W}_t^n, t \in [0,T]\}$, where $\hat{W}_t^n(h) = \sum_{j=1}^k \hat{W}^{n,j}(1_{[0,t]}h_j)$ for any $h \in \mathscr{H}^k$, and

$$\hat{W}^{n,j}(1_{[0,t]}h_j) = W^j(1_{[0,t]}h_j) + z_j \int_0^t \langle h_j(*), g_n^j(s,*)\rangle_{\mathscr{H}} ds.$$

Set the $k \times k$ matrix $\varphi_n^z(t,x) := \partial_z \hat{u}_n^z(t,x)$, and the Hessian matrix of the random vector $\hat{u}_n^z(t,x)$, $\psi_n^z(t,x) := \partial_z^2 \hat{u}_n^z(t,x)$ (which is a tensor of order 3). We denote by $\|\cdot\|$ the norm of a $n \times n$ matrix A defined as

$$\|A\| = \sup_{\xi\in\mathbb{R}^n, \|\xi\|=1} \|A\xi\|.$$

We next proceed to the study of the strict positivity of the density $p_{t,x}(\cdot)$ of the law of $u(t,x)$, where $(t,x) \in]0,T] \times \mathbb{R}^d$ are fixed. We need to introduce the following condition. We say that $y \in \mathbb{R}^k$ satisfies $\mathbf{H_{t,x}(y)}$ if

$\mathbf{H_{t,x}(y)}$ there exists a sequence of predictable processes $\{g_n\}_{n\geq 1}$ in $\mathscr{H}_T^k$, and positive constants c_1, c_2, r_0 and δ such that

(i) $\limsup_{n\to\infty} \mathrm{P}\left\{(\|u(t,x) - y\| \leq r) \cap (\det \varphi_n^0(t,x) \geq c_1)\right\} > 0$, for all $r \in]0, r_0]$.

(ii) $\lim_{n\to\infty} \mathrm{P}\left\{\sup_{\|z\|\leq\delta}(\|\varphi_n^z(t,x)\| + \|\psi_n^z(t,x)\|) \leq c_2\right\} = 1$.

Remark 5.5. Observe that if a point y verifies $\mathbf{H_{t,x}(y)}$ then automatically belongs to the support of the law of $u(t,x)$.

We next provide the main result of this section which is a criterion for the strict positivity of a density, which was proved in [1, Theorem 3.3] for the case where $\mathscr{H}_T^k$ is replaced by $L^2([0,1];\mathbb{R}^k)$. Our case follows along the same lines as theirs. This criterium uses the following quantitative version of the classical inverse function theorem. Let $B(x;r)$ denote the ball of $\mathbb{R}^k$ with center x and radius $r > 0$.

Lemma 5.6. [1, Lemma 3.2] and [17, Lemma 4.2.1] *Let $\Phi : \mathbb{R}^k \mapsto \mathbb{R}^k$ be a $\mathscr{C}^2$ mapping such that for some constants $\beta > 1$ and $\delta > 0$,*

$$|\det \Phi'(0)| \geq \frac{1}{\beta} \quad \text{and} \quad \sup_{\|z\|\leq\delta} (|\Phi(z)| + |\Phi'(z)| + |\Phi''(z)|) \leq \beta.$$

Then there exists constants $R \in]0,1]$ and $\alpha > 0$, such that Φ is a diffeomorphism from a neighborhood of 0 contained in the ball $B(0;R)$ onto the ball $B(\Phi(0);\alpha)$.

Theorem 5.7. *Let $(t,x) \in]0,T] \times \mathbb{R}^d$ and $y \in \mathbb{R}^k$ be such that $\mathbf{H_{t,x}(y)}$ holds true. Suppose that the law of random vector $u(t,x)$ has a continuous density $p_{t,x}$ in a neighborhood of y. Then $p_{t,x}(y) > 0$. Moreover, if $\mathbf{H_{t,x}(y)}$ holds on $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \Sigma$, with $\Sigma = \{y \in \mathbb{R}^k : \sigma_1(y), ..., \sigma_q(y)$ span $\mathbb{R}^k\}$, then for every connected subset $\tilde{\Sigma} \subset \Sigma$ such that $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \tilde{\Sigma}$ is non void, $p_{t,x}$ is a strictly positive function on $\tilde{\Sigma}$.*

Proof. Let $y \in \mathbb{R}^k$ satisfying $\mathbf{H_{t,x}(y)}$. Then, proceeding as in [1, Theorem 3.3], using Lemma 5.6 with $\Phi(z) = \hat{u}_n^z(t,x)$, one can show that there exists n sufficiently large such that for any continuous and bounded function $f : \mathbb{R}^k \mapsto \mathbb{R}_+$ it holds that

$$\mathrm{E}[f(u(t,x))] \geq \int_{B(y,\frac{\alpha}{2})} f(v)\theta_n^y(v)dv,$$

where $\alpha > 0$ is the parameter of Lemma 5.6 and θ_n^y is a strictly positive continuous function on $B(y, \frac{\alpha}{2})$. This proves the first statement of the theorem.

We next assume that $\mathbf{H_{t,x}(y)}$ holds on $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \Sigma$. It suffices to check that if $\tilde{\Sigma} \subset \Sigma$ is a connected component of Σ such that $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \tilde{\Sigma}$ is non void, then $\tilde{\Sigma} \subset \mathrm{Supp}(\mathrm{P}_{u(t,x)})$. Indeed, this implies that $\mathbf{H_{t,x}(y)}$ holds for all $y \in \tilde{\Sigma}$ and by the first part of the theorem one obtains that $p_{t,x}(y) > 0$ and the theorem is proved.

Suppose that $\tilde{\Sigma} \subset \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ is false. Then one may find $x_1 \in \tilde{\Sigma}$ such that $x_1 \notin \mathrm{Supp}(\mathrm{P}_{u(t,x)})$. Moreover, since $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \tilde{\Sigma}$ is non void one may find $x_2 \in \mathrm{Supp}(\mathrm{P}_{u(t,x)}) \cap \tilde{\Sigma}$. Now, since $\tilde{\Sigma}$ is connected, one can find a continuous curve $x(\lambda)$, $\lambda \in [0,1]$ contained in $\tilde{\Sigma}$ with $x(0) = x_2$ and $x(1) = x_1$. Take now $\lambda_* = \sup\{\lambda : x(\lambda) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})\}$. Since $x_1 = x(1) \notin \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and the complementary of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$ is an open set, it follows that $\lambda_* < 1$. Then we have a sequence $\lambda_n \uparrow \lambda_*$ such that $x(\lambda_n) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and we also know that $x(\lambda) \notin \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ for $\lambda_* < \lambda \leq 1$. This means that $x(\lambda_*)$ is on the boundary of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$, and since this set is closed, we conclude that $x(\lambda_*) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$. Since $x(\lambda_*)$ belongs also to $\tilde{\Sigma}$ we have that $p_{t,x}(x(\lambda_*)) > 0$. But this is contradictory with the fact that $x(\lambda_*)$ is on the boundary of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$. ☐

Remark 5.8. Observe that Theorem 5.7 says that if a specific point y_* is in the support of the law of $u(t,x)$ then $p_{t,x}$ is strictly positive on the connected component of Σ that contains y_*. However, this criterion based on the inverse function theorem does not give information about the support of the law. Nevertheless, in order to prove that $p_{t,x}(y) > 0$, we do not have to check that y itself belongs to the support, but only that the connected component of Σ which contains y has a non void intersection with the support, and in particular, if we have ellipticity everywhere, then $\Sigma = \mathbb{R}^k$ and so the above condition is automatically verified (see Remark 5.2).

Remark 5.9. There is a little mistake in page 56, lines 11-13 of [1]. Indeed, the support is a closed set and their set $\{\varphi \neq 0\}^d$ is open, thus they may not intersect, so it is not always possible to choose a point y in the boundary of the support and such that $\varphi(y) \neq 0$. For example, if $\{\varphi \neq 0\} = \{-1,1\}^c$ and the support is $[-1,1]$, in the boundary of the support one has that $\varphi = 0$.

5.2 Proof of Theorem 5.1

Let $(t,x) \in]0,T] \times \mathbb{R}^d$ be fixed. As explained at the beginning of Section 5.1, let $y \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ be fixed such that $\sigma_1(y), ..., \sigma_k(y)$ span $\mathbb{R}^k$. Consider the sequence of predictable processes $\{g_n\}_{n\geq 1}$ in $\mathscr{H}_T^k$, defined by

$$g_n^j(s,z) = v_n^{-1} 1_{[t-2^{-n},t]}(s)\Gamma(t-s,x-z), \;\; n \geq 1, \; j=1,...,k, \tag{5.4}$$

where

$$v_n := \int_0^{2^{-n}} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi) dr.$$

We are going to prove that assumptions (i) and (ii) of $\mathbf{H_{t,x}(y)}$ are satisfied for this sequence of predictable processes. Then, then second statement of Theorem 5.7 will give the conclusion of Theorem 5.1.

We start by some preliminary computations. We write

$$\varphi_{n,i,j}^z(t,x) = \int_0^t \langle D_r^{(j)}(\hat{u}_{n,i}^z(t,x)), g_n^j(r,*)\rangle_{\mathscr{H}} dr, \; 1 \leq i,j \leq k,$$

which follows since the processes defined by the left hand-side and right hand-side satisfy the same equation, and there is uniqueness of solution. Then by (5.2) and the stochastic differential equation satisfied by the derivative (4.1), we obtain that

$$\varphi_{n,i,j}^z(t,x) = \mathscr{A}_{n,i,j}^z(t,x) + \mathscr{B}_{n,i,j}^z(t,x) + \mathscr{C}_{n,i,j}^z(t,x),$$

where

$$\mathscr{A}_{n,i,j}^z(t,x) = v_n^{-1} \int_0^{2^{-n}} \langle \sigma_{ij}(\hat{u}_n^z(t-r,*))\Gamma(r,x-*), \Gamma(r,x-*)\rangle_{\mathscr{H}} dr,$$

$$\mathscr{B}_{n,i,j}^z(t,x) := \int_0^t \Bigg(\int_r^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) \sum_{\ell,m=1}^k \partial_m \sigma_{i\ell}(\hat{u}_n^z(s,y)) \times \langle D_r^{(j)}(\hat{u}_{n,m}^z(s,y)), g_n^j(r,*)\rangle_{\mathscr{H}} \hat{W}^{n,\ell}(ds,dy) \Bigg) dr,$$

$$\mathscr{C}_{n,i,j}^z(t,x) := \int_0^t \Bigg(\int_r^t \int_{\mathbb{R}^d} \sum_{m=1}^k \partial_m b_i(\hat{u}_n^z(s,x-y)) \times \langle D_r^{(j)}(\hat{u}_{n,m}^z(s,x-y)), g_n^j(r,*)\rangle_{\mathscr{H}} \Gamma(t-s,dy) ds \Bigg) dr.$$

Note that $g_n^j(r,*) = 1_{[t-2^{-n},t]}(r) g_n^j(r,*)$, and that $D_r(\hat{u}_{n,i}^z(s,y)) = 0$ if $s<r$. Hence, using

these facts and Fubini's theorem, it yields that

$$\mathscr{B}^z_{n,i,j}(t,x) = \int_{t-2^{-n}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) \sum_{\ell,m=1}^k \partial_k \sigma_{i\ell}(\hat{u}^z_n(s,y)) \\ \times \left(\int_{t-2^{-n}}^s \langle D^{(j)}_r(\hat{u}^z_{n,m}(s,y)), g^j_n(r,*)\rangle_{\mathscr{H}} dr \right) \hat{W}^{n,\ell}(ds,dy),$$

$$\mathscr{C}^z_{n,i,j}(t,x) = \int_{t-2^{-n}}^t \int_{\mathbb{R}^d} \sum_{m=1}^k \partial_m b_i(\hat{u}^z_n(s,x-y)) \\ \times \left(\int_{t-2^{-n}}^s \langle D^{(j)}_r(\hat{u}^z_{n,m}(s,x-y)), g^j_n(r,*)\rangle_{\mathscr{H}} dr \right) \Gamma(t-s,dy)ds.$$

Therefore, we have proved that

$$\varphi^z_{n,i,j}(t,x) = \mathscr{A}^z_{n,i,j}(t,x) + \mathscr{D}^z_{n,i,j}(t,x) + \mathscr{E}^z_{n,i,j}(t,x) + \mathscr{C}^z_{n,i,j}(t,x), \tag{5.5}$$

where

$$\mathscr{D}^z_{n,i,j}(t,x) := \int_{t-2^{-n}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) \sum_{\ell,m=1}^k \partial_m \sigma_{i\ell}(\hat{u}^z_n(s,y)) \varphi^z_{n,m,j}(s,y) W^\ell(ds,dy),$$

$$\mathscr{E}^z_{n,i,j}(t,x) := \sum_{\ell,m=1}^k z_\ell \int_{t-2^{-n}}^t \langle \Gamma(t-s,x-*)\partial_m \sigma_{i\ell}(\hat{u}^z_n(s,*))\varphi^z_{n,m,j}(s,*), g^\ell_n(s,*)\rangle_{\mathscr{H}} ds,$$

$$\mathscr{C}^z_{n,i,j}(t,x) = \int_{t-2^{-n}}^t \int_{\mathbb{R}^d} \sum_{m=1}^k \partial_m b_i(\hat{u}^z_n(s,x-y)) \varphi^z_{n,m,j}(s,x-y) \Gamma(t-s,dy)ds.$$

Next we study upper bounds for the p-moments of the four terms on the right hand side of $\varphi^z_{n,i,j}(t,x)$. For this, we assume that $\|z\| \le \delta$ for some $\delta > 0$.

First observe that since σ is bounded,

$$|\mathscr{A}^z_{n,i,j}(t,x)| \le K. \tag{5.6}$$

Appealing to [19, (3.11)] and using the fact that the partial derivatives of the coefficients of σ are bounded, we get that for all $p > 1$,

$$\mathrm{E}[|\mathscr{D}^z_{n,i,j}(t,x)|^p] \le c_p v_n^{p/2} \sum_{m=1}^k \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p]. \tag{5.7}$$

Using the Cauchy-Schwarz inequality and the fact that the derivatives of σ are bounded, it yields that for all $p > 1$,

$$\mathrm{E}[|\mathscr{E}^z_{n,i,j}(t,x)|^p] \le c_p \delta^p \sum_{m=1}^k \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p]. \tag{5.8}$$

We finally use Minkowski's inequality with respect to the finite measure $\Gamma(s,dy)ds$, the boundedness of the partial derivatives of the coefficients of b, and hypothesis (2.2), to see that for all $p>1$,

$$\begin{aligned}\mathrm{E}[|\mathscr{C}^z_{n,i,j}(t,x)|^p] &\le c_p \sum_{m=1}^k \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p]\Big(\int_0^{2^{-n}}\int_{\mathbb{R}^d}\Gamma(s,dy)ds\Big)^p \\ &\le c_{p,T}2^{-np}\sum_{m=1}^k \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p].\end{aligned} \tag{5.9}$$

Now, introducing (5.6), (5.7), (5.8) and (5.9) into (5.5), we get that for all $p>1$,

$$\sum_{m=1}^k \mathrm{E}[|\varphi^z_{n,m,j}(t,x)|^p] \le K_p + c_{p,T}(v_n^{p/2}+\delta^p+2^{-np})\sum_{m=1}^k \sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p]. \tag{5.10}$$

Observe that, proceeding as in the proof of (4.2), one can show that

$$\sup_{\|z\|\le\delta}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\mathrm{E}[|\varphi^z_{n,i,j}(s,y)|^p]<\infty, \tag{5.11}$$

as we have shown in (5.5) that $\varphi^z_{n,i,j}(t,x)$ satisfies a linear equation with initial condition $\mathscr{A}^z_{n,i,j}(t,x)$, which is bounded. Thus, choosing n large and δ small such that $c_{p,T}(v_n^{p/2}+\delta^p+2^{-np})\le\frac{1}{2}$, we obtain from (5.10) and (5.11) that

$$\sup_{\|z\|\le\delta}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\mathrm{E}[|\varphi^z_{n,i,j}(s,y)|^p]\le K_p. \tag{5.12}$$

We are now ready to show that (i) and (ii) of $\mathbf{H_{t,x}(y)}$ are verified.

Proof of (i). Take $z=0$ in (5.5), and use (5.7), (5.9) and (5.12), to get that

$$\varphi^0_{n,i,j}(t,x)=\mathscr{A}^0_{n,i,j}(t,x)+\mathscr{R}_{n,i,j}(t,x),$$

where, for any $p>1$,

$$\mathrm{E}[|\mathscr{R}_{n,i,j}(t,x)|^p]\le c_{p,T}(v_n^{p/2}+2^{-np}).$$

We now write

$$\mathscr{A}^0_{n,i,j}(t,x)=\sigma_{ij}(u(t,x))+\mathscr{O}_{n,i,j}(t,x),$$

where

$$\mathscr{O}_{n,i,j}(t,x)=v_n^{-1}\int_0^{2^{-n}}\Big\langle(\sigma_{ij}(u(t-r,*))-\sigma_{ij}(u(t,x)))\Gamma(r,x-*),\Gamma(r,x-*)\Big\rangle_{\mathscr{H}}dr.$$

By Lemma 4.2 and hypothesis **(H6)**, it holds that for all $p>1$,

$$\mathrm{E}[|\mathscr{O}_{n,i,j}(t,x)|^p]\le c_{p,T}2^{-n(\alpha-\eta)p}\quad(\alpha>\eta).$$

Now, as $y \in \text{Supp}(\mathrm{P}_{u(t,x)}) \cap \Sigma$, there exists $r_0 > 0$ such that for all $0 < r \leq r_0$,

$$B(y;r) \subset \Sigma, \quad \text{and} \quad \mathrm{P}\left\{u(t,x) \in B(y;r)\right\} > 0.$$

Moreover, $\sigma_1(y), ..., \sigma_k(y)$ span $\mathbb{R}^k$. Let $\sigma(y)$ denotes the matrix with columns $\sigma_1(y), ..., \sigma_k(y)$. Hence, for all $0 < r \leq r_0$,

$$\mathrm{P}\left\{(\|u(t,x) - y\| \leq r) \cap (\det \sigma(u(t,x)) \geq 2c_1)\right\} > 0,$$

where

$$c_1 := \frac{1}{2}\left(\inf_{z \in B(y;r)} \inf_{\|\xi\|=1} \|\sigma(z)\xi\|^2\right)^k.$$

Thus, we conclude that

$$\limsup_{n\to\infty} \mathrm{P}\left\{(\|u(t,x) - y\| \leq r) \cap (\det \varphi_n^0(t,x) \geq c_1)\right\} > 0,$$

which proves (i).

Proof of (ii). We start proving that there exist $c > 0$ and $\delta > 0$, such that

$$\lim_{n\to\infty} \mathrm{P}\left\{\sup_{\|z\|\leq\delta} \|\varphi_n^z(t,x)\| \leq c\right\} = 1. \tag{5.13}$$

Observe that (5.5), together with (5.6), (5.7), (5.8) and (5.9), show that

$$\|\varphi_n^z(t,x)\| \leq K + \sup_{\|z\|\leq\delta} \|\mathscr{G}_n^z(t,x)\|, \tag{5.14}$$

where for any $p > 1$,

$$\sup_{\|z\|\leq\delta} \mathrm{E}[\|\mathscr{G}_n^z(t,x)\|^p] \leq c_{p,T}(v_n^{p/2} + \delta^p + 2^{-np}).$$

Hence, in order to prove (5.13) one only needs to check the uniformity in z. Let z and z' such that $\|z\| \vee \|z'\| \leq \delta$. Then, using (5.5) and similar computations as in (5.7), (5.8) and (5.9), and appealing to the Lipschitz property of the derivatives of the coefficients of σ and b and (5.12), together with the Cauchy-Schwarz inequality, and finally choosing n large and δ small, we obtain that

$$\begin{aligned}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\left[\|\varphi_n^z(s,y) - \varphi_n^{z'}(s,y)\|^p\right] &\leq c_p \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\left[\|\mathscr{A}_n^z(s,y) - \mathscr{A}_n^{z'}(s,y)\|^p\right] \\ &\quad + C_{p,T} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\left[\|\hat{u}_n^z(s,y) - \hat{u}_n^{z'}(s,y)\|^{2p}\right]^{1/2}.\end{aligned}$$

We now claim that for all $p > 1$,

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\left[\|\hat{u}_n^z(s,y) - \hat{u}_n^{z'}(s,y)\|^p\right] \leq c_{p,T}\|z - z'\|^p. \tag{5.15}$$

Indeed, using the Lipschitz property of the coefficients of σ and b, together with [19, (3.9), (5.15)], we obtain that

$$
\begin{aligned}
\mathrm{E}\left[\|\hat{u}_n^z(t,x)-\hat{u}_n^{z'}(t,x)\|^p\right] &\le c_p\|z-z'\|^p \\
&+ c_{p,T}\int_0^t \sup_{y\in\mathbb{R}^d} \mathrm{E}\left[\|\hat{u}_n^z(s,y)-\hat{u}_n^{z'}(s,y)\|^p\right]\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t-s)(\xi)|^2\mu(d\xi)ds \\
&+ c_{p,T}\int_0^t \sup_{y\in\mathbb{R}^d} \mathrm{E}\left[\|\hat{u}_n^z(t-s,y)-\hat{u}_n^{z'}(t-s,y)\|^p\right]\int_{\mathbb{R}^d}\Gamma(s,dy)ds.
\end{aligned}
$$

Thus, hypotheses (2.2) and (2.1), and Gronwall's lemma prove (5.15).

We next use the Lipschitz property of σ together with (5.15), to obtain that

$$
\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\left[\|\mathscr{A}_n^z(s,y)-\mathscr{A}_n^{z'}(s,y)\|^p\right] \le c_{p,T}\|z-z'\|^p.
$$

Therefore, we have proved that

$$
\mathrm{E}\left[\|\varphi_n^z(t,x)-\varphi_n^{z'}(t,x)\|^p\right] \le c_{p,T}\|z-z'\|^p,
$$

which, together with (5.14) concludes the proof of (5.13).

The proof of (ii) for $\psi_n^z(t,x)$ follows along the same lines, therefore we only give the main steps. Let

$$
\psi_{n,i,j,m}^z(t,x) = \frac{\partial^2}{\partial z_m \partial z_j}\hat{u}_{n,i}^z(t,x).
$$

Then we have that

$$
\psi_{n,i,j,m}^z(t,x) = \int_0^t\int_0^t \langle D_s^{(m)}D_r^{(j)}(\hat{u}_{n,i}^z(t,x)), g_n^j(r,*)\otimes g_n^m(s,*)\rangle_{\mathscr{H}\otimes\mathscr{H}}drds,
$$

where the $\mathscr{H}\otimes\mathscr{H}$-valued process $D_s^{(m)}D_r^{(j)}(\hat{u}_{n,i}^z(t,x))$, satisfies the following linear stochastic differential equation

$$
\begin{aligned}
D_s^{(m)}D_r^{(j)}(\hat{u}_{n,i}^z(t,x)) &= \Gamma(t-r,x-*)D_s^{(m)}(\sigma_{ij}(u(r,*))) + \Gamma(t-s,x-*)D_r^{(j)}(\sigma_{im}(u(s,*))) \\
&+ \int_{r\vee s}^t\int_{\mathbb{R}^d}\Gamma(t-w,x-y)\sum_{\ell=1}^k D_s^{(m)}D_r^{(j)}(\sigma_{i\ell}(u(w,y)))W^\ell(dw,dy) \\
&+ \int_{r\vee s}^t\int_{\mathbb{R}^d}\Gamma(t-w,dy)D_s^{(m)}D_r^{(j)}(b_i(u(w,x-y)))\,dw.
\end{aligned}
$$

Using the chain rule and the stochastic differential equation satisfied by the first derivative, one can compute the different terms of $\psi_{n,i,j,m}^z(t,x)$ as we did for $\varphi_{n,i,j}^z(t,x)$, and bound their pth-moments. Finally, one estimates the pth-moments of the difference $\psi_n^z(t,x)-\psi_n^{z'}(t,x)$ as we did for $\varphi_n^z(t,x)$ in order to get the desired result.

5.3 Proof of Theorem 5.3

The existence and smoothness of the density follows from [19, Theorem 6.2]. Hence, we only need to prove the strict positivity. For this, one applies Theorem 5.7 as in Theorem 5.1 taking $\Sigma = \mathbb{R}$. In this case, in order to prove hypothesis (i), one proceeds as in Theorem 5.1, using hypotheses **(H1)**, to show that for all $p > 1$,

$$\lim_{n\to\infty} \mathrm{E}\left[|\varphi_n^0(t,x) - \mathscr{A}_n^0(t,x)|^p\right] = 0.$$

Next using the non-degeneracy assumption on σ, one gets that $\mathscr{A}_n^0(t,x) \geq c$, which implies that

$$\varphi_n^0(t,x) \geq c - |\varphi_n^0(t,x) - \mathscr{A}_n^0(t,x)|.$$

Finally, these assertions imply that for all $y \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$

$$\limsup_{n\to\infty} \mathrm{P}\left\{ (|u(t,x) - y| \leq r) \cap (\varphi_n^0(t,x) \geq \frac{c}{2}) \right\} > 0,$$

which proves (i). The proof of (ii) follows exactly as in Theorem 5.1.

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