ON THE JUSTICE OF VOTING SYSTEMS*

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Abstract. What are the best voting systems in terms of utilitarianism? Or in terms of maximin, or maximax? We study these questions for the case of three alternatives and a class of structurally equivalent voting rules. We show that plurality, arguably the most widely used voting system, performs very poorly in terms of remarkable ideals of justice, such as utilitarianism or maximin, and yet is optimal in terms of maximax. Utilitarianism is best approached by a voting system converging to the Borda count, while the best way to achieve maximin is by means of a voting system converging to negative voting. We study the robustness of our results across different social cultures, measures of performance, and population sizes.

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1. Introduction

Should I implement a plurality voting rule if I am a convinced utilitarian? If I wish to maximize an ideal of justice like maximin, should I implement the Borda count? This is the type of question addressed in this paper. Our aim is to evaluate voting systems in terms of how well they perform in relation to some particular ideal of justice.

Many different voting systems may be used for making group decisions involving the selection of an alternative from a given set. Obvious scenarios are the political arena, committee decisions (in firms, clubs, professional associations, universities, juries), or tournaments where a winner must be selected. It appears therefore that a proper understanding of the characteristics of voting systems is of great importance.

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We are certainly not the first in evaluating voting systems. As long ago as 1958, Black proposed a criterion that has attracted a great deal of attention. This criterion ranks voting systems by the probability of their selecting the Condorcet winner. That is, in terms of the probability of their leading to the alternative that is preferred over any other by a simple majority of voters. The literature has followed this line of inquiry and has contrasted voting systems on the basis of other criteria such as strategy-proofness, consistency of the social preference ordering, Pareto-dominance, path-independence, complexity, etc.[1]

In this paper, we focus not on the specific properties that voting systems may or may not satisfy, but on how well they perform in terms of an ideal of justice. If there is agreement over which ideal of justice is to be maximized, it seems much more relevant to evaluate voting systems in terms of how well they serve that ideal of justice, than for example it is to calculate the probability of their electing the Condorcet winner. In other words, in this paper we shift the evaluation criterion for voting systems from the ordinal to the cardinal approach.

In our setting, agents have cardinal and interpersonally comparable utility functions over the set of alternatives. The realization of the individual utility values depend on the culture of the society. We start the analysis by assuming that for every individual, the possible utility values associated to the different alternatives follow a uniform distribution. That is, we use a cardinal version of the well-known impartial culture condition.

We study the class of voting systems that encompasses the set of scoring rules that aggregate the strength or intensity of individuals' vote for each of three alternatives. Intensity is contingent on the position of that alternative in the ranking given by the individual. That is, a scoring rule is defined by the values an individual may assign to the best, middle, and worst alternatives in the reported ranking. We call such a set of scoring rules *fixed-value scoring rules.* Notice that this class of scoring rules includes plurality voting where one person has one vote, the Borda count, negative voting where voters name their worst alternative, and any combination of the above. In section 4 we offer a method to find the set of all different scoring rules when there are n agents

[1]Early studies are Brams and Fishburn (1978), Caplin and Nalebuff (1988), DeMeyer and Plott (1970), and Nurmi (1983). For an extensive survey on the evaluation of voting rules in terms of the probability of their electing the Condorcet winner see Gehrlein (1997). See also Lepelley, Pierron, and Valognes (2000), Levin and Nalebuff (1995), and Myerson (2002).

and 3 alternatives.[2] Throughout this paper, we will assume that individuals honestly report their true ranking of alternatives.

We evaluate the performance of a scoring rule for a given theory of justice on the basis of the probability of its selecting the best alternative for that theory of justice. For the case of two voters and three alternatives we can offer the mapping between any combination of maximin and maximax (including utilitarianism) and the ranking of scoring rules. It emerges that plurality, probably the most widely used voting system, maximizes an ideal of justice like maximax, being the worst scoring rule in terms of utilitarianism and maximin. That is, instead of maximizing efficiency or the well-being of the worst-off agents in a society, *plurality maximizes the well-being of the best-off*! Furthermore, utilitarianism is best approached by a scoring rule that falls in between plurality and Borda, while a scoring rule between Borda and negative is the best for maximin.

We then extend the analysis to more general social cultures, capturing different degrees of extremism in the evaluation of alternatives. That is, we study a complete class of symmetric density functions defined over the closed unit interval. Later, we also explore an alternative means to measure the performance of different scoring rules for different ideals of justice. In particular, instead of focusing solely on the probability of selecting the best alternative for a particular ideal of justice, we incorporate a method to measure the magnitude of the deviations from it. The results of the analysis of these extensions show a great deal of robustness with respect to our initial conclusions. The main results highlighted above remain the same. We then turn to the study of the general case of $n \times 3$. We offer analytical results and insights for the different ideals of justice and we complete the study via a simulation model. Remarkably, all these results for larger societies again largely reproduce the basic results obtained in the 2×3 case. In a nutshell, we show that plurality is still one of the best scoring rules for maximax. In fact, we show that as n tends to infinity, plurality is the best scoring rule in terms of maximax. Plurality is easily outperformed by other scoring rules for the pursuit of either utilitarian or maximin principles. Maximin is best approached by scoring rules that tend towards negative voting. Finally, utilitarianism is generally best approached by scoring rules that tend towards Borda.

There is very little work on evaluating voting systems on the basis of some theory of justice. Notable exceptions are the early simulation studies of Bordley (1983) and Merrill (1984). Bordley and Merrill analyze different voting

[2] We say that two scoring rules are different whenever there are instances in which they differ in their choice of best alternative.

systems, including plurality and Borda, in terms of their efficiency by using simulations. Consistent with our results, they show that plurality is easily outperformed in utilitarian terms. Our work differs from theirs, as they only consider utilitarianism as the sole justice criterion, while we consider general social welfare functions as ideals of justice. Furthermore, our approach is primarily theoretical. Finally, the range of voting systems under scrutiny also differs: while they study some notable procedures (such as Copeland system, the Hare system, or approval voting), we focus on a set of structurally equivalent rules, i.e., the set of all possible fixed-value scoring rules.

In another related strand of literature, there are papers that study how to choose a voting rule in a constitutional setting where there are two options, the status quo and a second alternative, and individual preferences are uncertain. A voting rule is characterized by the number of votes needed to accept the second alternative over the status quo. Papers in this literature study which voting rules maximize efficiency, which are self-stable, how to weight votes in heterogenous contexts, self-enforcement voting rules, etc. In this line, see Rae (1969), Barbera and Jackson (2004, 2006), Maggi and Morelli (2006) and papers cited therein.[3]

The remainder of the paper is organized as follows. Section 2 introduces the notation and the basic definitions that will be used subsequently. Section 3 contains the first theoretical analysis of the 2×3 case. Section 4 reports the extensions to the former analysis, and a method for finding the set of all different scoring rules. Finally, Section 5 presents some conclusions. All the proofs are given in the appendix.

2. Preliminaries

Let N be a finite set of individuals with cardinality n and $K = \{1, 2, 3\}$ the set of alternatives. A society U is a matrix of n rows and 3 columns with entries u_{il} in $[0, 1]$, denoting cardinal and interpersonally comparable utility values. Then, u_{il} is the utility value for individual i with respect to alternative l. A culture C is a probability distribution over the set of all possible societies $\mathcal{U} = [0, 1]^{n \times 3}$. We start the analysis by studying cultures where for every individual, the possible utility values associated with the different alternatives follow the uniform distribution. That is, we start by assuming the:

[3] Alternatively, there are papers that study the different properties of voting rules in the context of the provision of a public good (see, e.g., Aghion and Bolton (2003), Harstad (2005), and Lizzeri and Persico (2001)).

Impartial Culture Condition (ICC): C is the uniform probability distribution over $\mathcal{U}$.

Note that ICC is equivalent to considering u_{il} as an independent uniform realization from $[0,1]$ for every i and l. This is a cardinal version of the well-known and extensively studied property that states that all preference orders are assumed equally likely.

A Social Welfare Function[4] SWF is a mapping W from $[0,1]^n$ to $[0,1]$. That is, let $u_l = (u_{1l}, \ldots, u_{nl}) \in [0,1]^n$, then W aggregates the individual utility values in u_l into a social utility. $W_{Best}(U)$ denotes an alternative $l \in K$ that maximizes W in $U = (u_1, u_2, u_3)$. Prominent SWFs in the literature of distributive justice are utilitarianism and maximin. As it is well-known, *utilitarianism* takes the sum of individual utilities to evaluate an alternative. Formally, a SWF is utilitarian if $W = W^{Ut}$ with $W^{Ut}(u_l) = \sum_{i=1}^{i=n} u_{il}$. Therefore, $W^{Ut}_{Best}(U) \in \text{argmax}_l W^{Ut}(u_l)$.[5] The maximin principle disregards the utility values of the most favored individuals to evaluate an alternative on the basis of the utility value of the worst-off agent. In other words, a SWF is of the maximin type if $W = W^{Min}$ with $W^{Min}(u_l) = \min\{u_{1l}, \ldots, u_{nl}\}$, and $W^{Min}_{Best}(U) \in \text{argmax}_l W^{Min}(u_l)$. Consider also the *maximax* rule. Contrary to maximin, the maximax rule focuses on the best-off individuals. That is, a SWF is of the maximax type if $W = W^{Max}$ with $W^{Max}(u_l) = \max\{u_{1l}, \ldots, u_{nl}\}$, and $W^{Max}_{Best}(U) \in \text{argmax}_l W^{Max}(u_l)$. Maximax as an ideal of justice may appear just as a formal curiosity. However, it is widely used in professional sports to select the winner of a contest. More importantly, we will see that the maximax principle is much more common in democratic political institutions that one might expect.

A society U in ordinal terms U_O is the collection of n complete preorders $(\succeq_1, \ldots, \succeq_n)$ over K which are consistent with the rows of U. That is, for any $l, h \in K$ and $i \in N$, $l \succeq_i h$ if and only if $u_{il} \geq u_{ih}$.[6] Denote by $\mathcal{U}_O$ the set of all possible societies in ordinal terms.

A Scoring Rule S can be represented by a vector $S = (S_1, S_2, S_3) \in [0,1]^3$ denoting the strength of an individual's vote for her most, middle, and least preferred alternatives. We normalize strength of vote by assigning a value of 1 to the best reported alternative, and a value of 0 to the worst. Note that this

[4]Throughout the paper we use the terms "social welfare function" and "ideal of justice" indistinctly.

[5]In the event of a tie, we assume that an alternative is chosen randomly among those with the highest values, each with equal probability.

[6]Note that due to the continuity of $\mathcal{U}$ and the ICC, the probability of a tie in U equals zero, and hence $(\succeq_1, \ldots, \succeq_n)$ can be regarded as linear orders.

normalization of the extreme values is without loss of generality. Therefore, in this context a scoring rule is characterized by the value $S \in [0,1]$ assigned to the middle alternative. Then, plurality voting is identified with $S = 0$, the Borda count with $S = 1/2$, and negative voting with $S = 1$. We say that a rule is an implosion scoring rule if $S \in (0, 1/2)$, while if $S \in (1/2, 1)$ it is an explosion rule.[7] We will assume that for every individual i better alternatives in terms of $\succeq_i$ are assigned more points. That is, we assume honest voting. Now, the score of an alternative is the sum of points it is assigned across all individuals. $S_{Best}(U_O)$ is any alternative that has received the highest number of points. For a given population n, two scoring rules S_1 and S_2 are different if there are instances in which they differ in their choice of best alternative. The number of different scoring rules depends on n. Clearly, if $n = 2$ there are 5 different scoring rules, since every $S \in (0, 1/2)$ represents the same scoring rule, and similarly, every $S \in (1/2, 1)$ also represents the same scoring rule. Hence, the set of different scoring rules when $n = 2$ compromises plurality ($S = 0$), Borda ($S = 1/2$), negative ($S = 1$), an explosion rule (with any value $S \in (1/2, 1)$), and an implosion rule (with any value $S \in (0, 1/2)$). We analyze the set of different scoring rules contingent upon n in section 4.

It is our aim to contrast different scoring rules on the basis of their performance in relation to certain theories of justice. A natural index, which we denote by $I(C, W, S)$, measures the probability of a scoring rule S selecting the optimal alternative for a SWF W, given a culture C, i.e.

$$I(C, W, S) = p(U : W_{Best}(U) = S_{Best}(U_O))$$

This index has the advantage of following the same criterion to evaluate all possible ideals of justice, and hence comparability of the success of a given scoring rule across ideals of justice is possible. Alternative measures are considered in section 4.

3. Characterization of a Small Society: The 2×3 Case

We begin the analysis by studying the case of two individuals and three alternatives. Recall that in such a setting there are 5 different classes of scoring rules: (i) plurality voting with $S = 0$, (ii) Borda count with $S = 1/2$, (iii) negative voting with $S = 1$, (iv) explosion rules with any value of S in the interval $(1/2, 1)$, and finally (v) implosion rules with any value of S in $(0, 1/2)$.

[7] Alternatively one could call these worst-punishing rules (best-rewarding in the case of implosion rules) as in Cox (1990) or Myerson (2002). However, this terminology would also include negative (respectively, plurality) and hence, we prefer to avoid it.

Our first result provides an ordering of scoring rules contingent upon the three ideals of justice presented in the previous section: utilitarianism, maximin and maximax.

Theorem 3.1. *Let $n = 2$, $k = 3$, and ICC hold, then the rankings of the scoring rules according to the index I (with values in parenthesis) are:*[8]

- ***Utilitarianism**: implosion (.743), Borda (.739), explosion (.733), negative (.65), and plurality (.642).*
- ***Maximin**: explosion (.783), Borda (.739), implosion (.717), negative (.7), and plurality (.617).*
- ***Maximax**: plurality (.667), implosion (.667), Borda (.639), explosion (.583), and negative (.5).*

There are several noteworthy remarks to make in relation to Theorem 3.1. First, Theorem 3.1 gives a clear picture of how to select among scoring rules according to a criterion of justice. If one wants to maximize utilitarianism, then one should implement an implosion rule. If it is maximin the principle that one wants to maximize, then an explosion rule is required. Finally, maximax is best approached by plurality or by an implosion rule.

Second, note that plurality, probably the most widely used scoring rule, performs optimally in terms of maximax. At the same time plurality is the worst scoring rule both in terms of utilitarianism and maximin. That is, instead of maximizing efficiency or the well-being of the worst-off agents in a society, *plurality maximizes the well-being of the best-off.*

Third, note that maximin and maximax are not symmetric. In particular, while the best scoring rules for maximax are either plurality or implosion, in the case of maximin the only scoring rule that maximizes the consistency index is the symmetric of implosion (i.e., explosion).

Fourth, differences in the performance of scoring across SWFs are due to the existence of what we call type 4 and 5 societies (see the proof of Theorem 3.1). A type 4 society is characterized by individuals agreeing on the alternative that they regard as middle, but disagreeing on the rest (the worst alternative for one agent is the best for the other, and viceversa).[9] In a type 5 society there is no agreement whatsoever between agents on a given position. We will study these two types of societies in detail below.

Finally, note that in utilitarianism, whenever the utility values of the two agents are equally weighted, implosion rules are better than Borda, and Borda

[8]The values shown are approximations. The exact values are reported in the Appendix.

[9]This class of societies has attracted a good deal of attention. Interestingly, Börgers and Postl (2005) study for this class of societies whether there are incentive compatible rules which elicit utilities and implement efficient decisions.

better than explosion rules. In maximin, however, where the only utility value that counts is the lowest (thus the highest utility value is assigned a weight of zero, while the lowest is assigned a weight of, say, 1), the order is reversed: first explosion, then Borda, and finally implosion. This result suggests the possibility of a continuous relation between the weighting of individual utility values and the order of these scoring rules. That is, between the ideal of justice and the ranking of scoring rules. To explore this and other issues, we study the following formulation of a SWF that generalizes the three already considered.

For each alternative l, the lowest individual utility value is weighted by $\lambda \in [0, 1]$, while the highest utility value is weighted by $(1 - \lambda)$. Denote this SWF by $G(\lambda)$. Then,

$$G(\lambda, u_l) = (1 - \lambda) \cdot max\{u_{il}, u_{jl}\} + \lambda \cdot min\{u_{il}, u_{jl}\}$$

and $G_{Best}(\lambda, U) \in \ \text{argmax}_l G(\lambda, u_l)$. Clearly, $G(1/2)$ corresponds to utilitarianism, $G(1)$ to maximin, and $G(0)$ to maximax. Hence, $G(\lambda)$ includes any possible compromise between these ideals of justice. For example, one may entertain the notion that, while neither utilitarianism, nor maximin are completely compelling, some combination of the two may be. Consequently, there exists a value of λ in the $(1/2, 1)$ interval that satisfies one's aspirations of justice.

Therefore, we can embed the three SWFs in $G(\lambda)$ and study any possible combination of them. To this end, the only types of societies we need to analyze are types 4 and 5. It is easy to see that for every $\lambda \in [0, 1]$ the performance of the scoring rules in societies of types 1, 2, and 3 coincide with that obtained when $\lambda \in \{0, 1/2, 1\}$ (see the proof of Theorem 3.1). For type 4 societies consider, without loss of generality, that alternative 1 is regarded as best and worst by agents i and j, alternative 2 is regarded by both players as the middle alternative, and hence alternative 3 is regarded as worst and best. Now, for $l \in \{1, 2, 3\}$ let

$$r_l = max\{u_{il}, u_{jl}\} \text{ and } p_l = min\{u_{il}, u_{jl}\}$$

That is, r_l denotes the utility value of the agent that values l most, while p_l denotes the utility value of the agent that values alternative l least. Then, the probability of $G(\lambda)$ selecting alternative 2 when $\lambda \in [0, 1] \setminus \{0, 1/2, 1\}$[10] is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H_1}} \int_{\mathbf{H_2}} f(r_1, r_2, r_3, p_1, p_2, p_3) dr_1 dr_3 dr_2 dp_2 dp_1 dp_2 \tag{3.1}$$

[10]We exclude values $\lambda = 0, 1/2, 1$ since our evaluation of function (3.1) implies indeterminacies at these values. For more on this issue see below.

where $f(r_1, r_2, r_3, p_1, p_2, p_3)$ is the joint-density function of the continuous random variables $u_{i1} > u_{i2} > u_{i3}$ and $u_{j1} < u_{j2} < u_{j3}$, and H_1 and H_2 are the sets $H_1 = \{t : t \leq r_2 + \frac{\lambda}{(1-\lambda)}(p_2 - p_3)\}$ and $H_2 = \{t : t \leq r_2 + \frac{\lambda}{(1-\lambda)}(p_2 - p_1)\}$.

Expression (3.1) is complex to solve. Not only does it have to identify the best-off individuals in the three alternatives, it also has to satisfy a number of conditional requirements. The best software packages available to us were not able to solve it. Therefore, we solve it by decomposing it in a number of cases. We omit the tedious details here but report them in a supplement to this paper.[11] Then, we obtain that function (3.1) is equal to:

$$\frac{5\lambda + 3 \cdot \lambda^2 - 2 \cdot \lambda^3}{10} \tag{3.2}$$

Note that when $\lambda = 0, 1/2, 1$, (3.2) is equal to $0, .3, 3/5$, as we obtained in the proof of Theorem 3.1. Hence, we conclude that function (3.1) is continuous in $[0, 1]$.

Now we turn to the case of type 5 societies. Without loss of generality, let alternative 1 be evaluated as best and middle, alternative 2 as middle and worst, and alternative 3 as worst and best by agents i and j respectively. Note that for every $\lambda \in [0, 1]$, $G(\lambda)$ never selects alternative 2. Then we can formulate the probability of $G(\lambda)$ selecting alternative 1 when $\lambda \in [0, 1] \setminus \{0, 1/2, 1\}$[12] as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H}} f(r_1, r_2, r_3, p_1, p_2, p_3) dr_1 dr_2 dr_3 dp_1 dp_2 dp_3 \tag{3.3}$$

with $H = \{t : t \geq r_3 + \frac{\lambda}{(1-\lambda)}(p_3 - p_1)\}$. As in (3.1), we had to decompose (3.3) in a number of cases until it could be solvable by standard computer resources. The details of the decomposition are contained in the supplement. Then, we obtain that (3.3) is equal to

$$\frac{1}{2} + \lambda - \frac{9\lambda^2}{10} + \frac{\lambda^3}{5} \tag{3.4}$$

and, as before, we also obtain that (3.3) is a continuous function in $[0, 1]$.

We are now in a position to obtain the exact values of the consistency index I for each of the scoring rules contingent on the value of λ. That is, for every possible combination of the utilitarian, maximin, and maximax SWFs, Theorem 3.2 offers the exact consistency values obtained by each of the possible scoring rules. Hence, for any possible SWF λ, Theorem 3.2 gives the necessary information to select the best scoring rule.

[11]The supplement is available in the webpages indicated in the opening of this paper.

[12]For the same reason as in function (3.1) values $\lambda = 0, 1/2, 1$ are omitted.

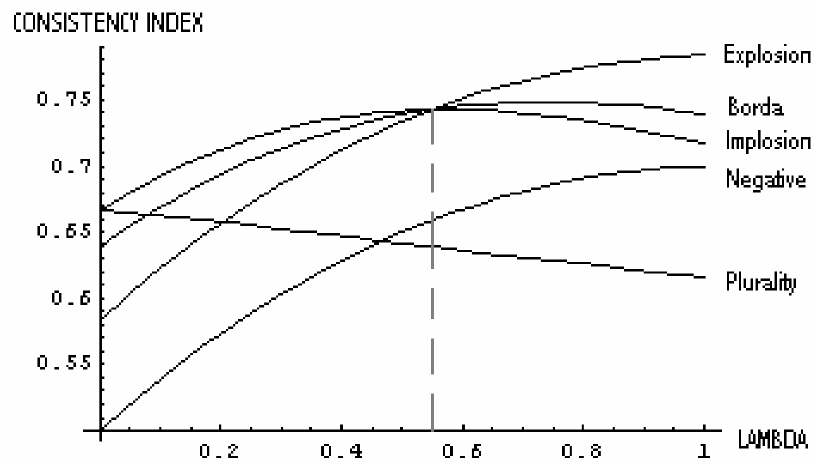


FIGURE 1. Results of Theorem 3.2

Theorem 3.2. *Let $n = 2$, $k = 3$, and ICC hold, then the values of the consistency index I in terms of $G(\lambda)$ are:*

- *Plurality:* $(80 - 5\lambda - 3\lambda^2 + 2\lambda^3)/120$,
- *Implosion:* $(80 + 35\lambda - 39\lambda^2 + 10\lambda^3)/120$,
- *Borda:* $(115 + 60\lambda - 54\lambda^2 + 12\lambda^3)/180$,
- *Explosion:* $(35 + 25\lambda - 15\lambda^2 + 2\lambda^3)/60$, *and*
- *Negative:* $(30 + 25\lambda - 15\lambda^2 + 2\lambda^3)/60$.

The above results can be summarized as follows. The consistency index I in the case of plurality is decreasing in λ, while in the cases of explosion and negative it is increasing. For implosion and Borda the index first increases and then decreases. Figure 1 shows the result graphically.

We end the analysis of the 2×3 case by studying the relation between the value of λ and the ranking of Borda, implosion and explosion. In other words, the relation between the criteria of justice λ and the best scoring rules in terms of $G(\lambda)$. Recall that when $\lambda \in \{0, 1/2\}$, implosion rules are better than Borda, and Borda better than explosion rules. However, when $\lambda = 1$ the order is reversed: first explosion, then Borda, and finally implosion. In Theorem 3.3

below, we show that there exists a value x

$$x = \frac{1}{2} + \sqrt{\frac{13}{3}} \sin(\frac{1}{3} \arcsin(\frac{2\sqrt{\frac{3}{13}}}{13}) \simeq .551$$

that defines a partition over the unit interval, offering the complete mapping between λ (the social welfare function) and the best scoring rules.

Theorem 3.3. *Let $n = 2$, $k = 3$, and ICC hold, then the scoring rules with the highest consistency index in terms of $G(\lambda)$ are:*

- *for $\lambda = 0$, plurality and implosion,*
- *for every $\lambda \in (0, x)$, implosion,*
- *for $\lambda = x$, Borda, implosion, and explosion, and*
- *for every $\lambda \in (x, 1]$, explosion.*

Theorem 3.3 shows that for values of λ lower than x implosion rules are better than Borda, and Borda better than explosion rules, for values of λ higher than x the order is reversed, and for $\lambda = x$ the three scoring rules obtain exactly the same consistency index. This, together with the previous results, allows us to give the mapping between λ and the best scoring rules.

4. Extensions

In this section we evaluate the robustness of the results we have obtained so far, by exploring alternative versions of some of our previous assumptions. In particular, (i) we check cultures other than the uniform one to explore the effect of different degrees of extremism, (ii) we develop an alternative index of adequacy between ideals of justice and scoring rules, and (iii) finally we study larger societies. For the latter, we obtain the set of different scoring rules contingent upon n.

4.1. **Alternative Probability Distributions.** We now study alternative cultures, i.e. probability distributions other than the uniform one. In particular, we study a complete class of symmetric density functions defined over the closed unit interval. This class of density functions aims to represent the case of polarized societies, in which extreme opinions prevail, and the case of homogenized societies with high levels of consensus. To this end, take the parabolic function $f(x) = \alpha x^2 + \beta x + \gamma$, with $x \in [0, 1]$. Now, in order for $f(x)$ to represent a symmetric density function, it must be the case that $\alpha = -\beta = -(1 - \gamma)$, with γ taking values in $[0, 3]$. Note that the $\gamma = 1$ case corresponds to the uniform distribution. Values of $\gamma \in [0, 1)$ correspond to strictly concave density functions. That is, lower values of γ represent higher levels of consensus on the evaluation of an alternative. Finally, values of $\gamma \in (1, 3]$ correspond to strictly convex density functions. Hence, higher values of γ represent higher levels of extremism. Note, then, that a density function in this class is characterized by its value γ.

In order to obtain the consistency indices between the three prominent ideals of justice (utilitarianism, maximin, and maximax) and the five different scoring rules in the 2×3 case, under any density function γ, one only has to follow the proof of Theorem 3.1 and correct for the required joint density function. We may conclude that very little changes. The consistency indices for maximin and maximax do not depend on γ, and hence coincide with those we obtained in Theorem 3.1. Those for utilitarianism do depend on the precise value of γ, but for every $\gamma \in [0, 3]$ the ranking of the scoring rules coincides with the one obtained under the uniform distribution.

Theorem 4.1. *Let $n = 2$, $k = 3$, then for every $\gamma \in [0, 3]$ the values of the consistency index I with respect to maximin and maximax are those obtained in Theorem 3.1. For utilitarianism are:*

- *Plurality:* $(3860816 - 11310\gamma^2 + 5760\gamma^3 - 1715\gamma^4 + 319\gamma^5 - 20\gamma^6)/6006000$,
- *Implosion:* $(4454936 + 13710\gamma^2 - 18800\gamma^3 + 4745\gamma^4 - 121\gamma^5 - 20\gamma^6)/6006000$,
- *Borda:* $(664693 + 3753\gamma^2 - 3684\gamma^3 + 969\gamma^4 - 66\gamma^5)/900900$,
- *Explosion:* $(2191994 + 23820\gamma^2 - 18040\gamma^3 + 4945\gamma^4 - 539\gamma^5 + 20\gamma^6)/3003000$,
- *Negative:* $(1941744 + 23820\gamma^2 - 18040\gamma^3 + 4945\gamma^4 - 539\gamma^5 + 20\gamma^6)/3003000$

In particular, for every $\gamma \in [0, 3]$ the ranking of scoring rules in terms of utilitarianism coincides with the one obtained in Theorem 3.1.

It is therefore important to stress that the above result gives a much greater degree of confidence in the robustness of the findings obtained in the uniform distribution.

4.2. **Reconsidering the Consistency Index.** Returning to the case of the uniform distribution, we now explore new indices for measuring the adequacy of the scoring rules to the different ideals of justice. Recall that the consistency index I measures the area in which a scoring rule coincides with an ideal of justice. This index says nothing about what happens in societies outside this area, however. In particular, it ignores the magnitude of errors, an issue on which we will now expand. For an intuitive sense of the nature of this index, we now propose to consider the following example for two individuals and three alternatives.

$$\mathcal{U} = \begin{pmatrix} 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & 0.8 \end{pmatrix}$$

In terms of utilitarianism the best alternative is alternative 3, with an associated social value of .6. Note that the utilitarian social value is the average[13] of the two respective individual values. Now consider a scoring rule, e.g. negative. Negative selects alternative 2 with an associated utilitarian social value of .55. Note that although negative does not select the utilitarian alternative, it does not select the worst either. Notice that the worst alternative for utilitarianism is alternative 1 with an associated utilitarian value of .5. Then the relative distance between utilitarianism and negative is $\frac{.55-.5}{.6-.5} = \frac{1}{2}$, that represents the relative loss (with respect to the worst possible performance) associated with the choice of alternative 2 rather than the utilitarian alternative 3. This is the kind of index addressed herein. Consider now the case of maximin. The best alternative for maximin is alternative 2, which has an associated social value of .5. Note that for a given alternative the maximin social value is different from the utilitarian social value. In the case of maximin it coincides with the utility value of the worst-off individual. Now an analogous analysis to that of the utilitarian case would give the relative distance between maximin and a given scoring rule.

One of the advantages of the original consistency index I is that it allows for comparisons between scoring rules across different ideals of justice. As the above example shows, this is no longer the case when considering relative measures of distance, since now the measure of performance now changes with each ideal of justice. Hence, comparability across ideals of justice is meaningless. However, if one is clear about the ideal of justice that should be implemented, then this new index may hold some attraction. We now turn to the formal definitions.

Recall that $G_{Best}(\lambda, U)$ is the best alternative in society U in terms of the ideal of justice λ. Then $G(\lambda, u_{G_{Best}(\lambda,U)})$ is the maximum social utility according to the ideal of justice λ in a society U. Also recall that $S_{Best}(U_O)$ denotes the best alternative in society U according to the scoring rule S. Then the associated social utility value of this alternative in terms of λ is $G(\lambda, u_{S_{Best}(U_O)})$. Thus, we are concerned here with the relative distance between $G(\lambda, u_{G_{Best}(\lambda,U)})$ and $G(\lambda, u_{S_{Best}(U_O)})$ across all possible societies. To this end consider the worst possible election in a society U, $G_{Worst}(\lambda, U) \in \text{argmin}_l G(\lambda, u_l)$. Hence, we study

(4.1)
$$\frac{G(\lambda, u_{S_{Best}(U_O)}) - G(\lambda, u_{G_{Worst}(\lambda,U)})}{G(\lambda, u_{G_{Best}(\lambda,U)}) - G(\lambda, u_{G_{Worst}(\lambda,U)})} = 1 - \frac{G(\lambda, u_{G_{Best}(\lambda,U)}) - G(\lambda, u_{S_{Best}(U_O)})}{G(\lambda, u_{G_{Best}(\lambda,U)}) - G(\lambda, u_{G_{Worst}(\lambda,U)})}$$

[13]Utilitarianism may also be expressed with the sum of utilities of both agents, without any effect on the results.

over all possible societies. That is, we compute the integral of (4.1) with respect to the culture distribution. We denote this index as $E(C, W, S)$.

In what follows, we focus on the uniform distribution and the cases of $\lambda \in \{0, 1/2, 1\}$.

Theorem 4.2. *Let $n = 2$, $k = 3$, and ICC hold, then the rankings of the scoring rules according to the index E (with values in parenthesis) are*[14]*:*

- ***Utilitarianism**: implosion (.917), Borda (.917), explosion (.917), negative (.857), and plurality (.857).*
- ***Maximin**: explosion (.914), Borda (.878), implosion (.860), negative (.857), and plurality (.768).*
- ***Maximax**: implosion (.869), plurality (.857), Borda (.851), explosion (.815), and negative (.768).*

There is a great deal of consistency between the results here and those obtained in Theorem 3.1. Notice that with respect to utilitarianism, implosion and explosion −and hence Borda− obtain the same value for E. Recall now that implosion was outperforming explosion when using the index I, i.e. when measuring the probability of choosing the best alternative. In other words, implosion selects the utilitarian alternative more often than explosion does. Then, since the two perform equally in terms of E, it must be the case that, when not selecting the best alternative, the loss of implosion is greater than the loss of explosion. With respect to maximin, nothing relevant changes. For maximax note that implosion is now the best scoring rule.

4.3. **Classes of Scoring Rules in n-agent Societies.** We start the analysis of the case of 3 alternatives and n individuals by studying the set of different scoring rules. Recall that we say that two scoring rules are different whenever there are profiles of societies (with positive measure) in which they result in different choices with respect to the best alternative.

Now, note that all the relevant information can be captured in a pair of two-dimensional vectors

$$B = (B^1, B^2), M = (M^1, M^2) \in \mathbb{N}^2$$

where B^l represents the number of people voting for alternative l as the best one, M^l represents the number of people voting for alternative l as the middle one, $l \in \{1, 2\}$, and $\mathbb{N}$ denotes the set of non-negative integers. It is straightforward to see that from this pair of vectors we can compute the number of people voting for any alternative $l \in \{1, 2, 3\}$ as best, middle, or worst.

[14]The values shown are approximations. The exact values are reported in the Appendix.

Recall that in the present context a scoring rule is characterized by the value $S \in [0,1]$ assigned to the middle alternative. Then, we first find those scoring rules S for which there exists a profile of preferences where:

- at least two alternatives (w.l.o.g, alternatives 1 and 2) are selected, and
- $B^1 > B^2$ (and consequently $M^1 < M^2$).

That is, alternatives 1 and 2 tie as top elements and this is not due to the fact that they have the same number of votes as the best and middle alternatives. The ordering $B^1 > B^2$ is without loss of generality. The following system of inequalities makes these two conditions compatible with all the restrictions on positivity we have to incorporate:

$$\begin{gathered} B^1 + S \cdot M^1 = B^2 + S \cdot M^2 \geq (n - B^1 - B^2) + S(n - M^1 - M^2) \\ B^1 > B^2, M^1 < M^2 \\ B^1 + M^1 \leq n, B^2 + M^2 \leq n \\ B^1 + B^2 \leq n, M^1 + M^2 \leq n \\ B^1 + B^2 + M^1 + M^2 \geq n \end{gathered} \tag{4.2}$$

Denote the set of solutions of such a system,[15] plus values 0 and 1 as set $E(n)$, and consider that $E(n) = \{e_0, e_1, \ldots, e_{Q(n)}\}$, with $e_0 = 0$ and $e_{Q(n)} = 1$, is ordered by the natural order relation $<$ on the rational numbers. For each pair e_q, e_{q+1} of consecutive values in $E(n)$, $q \in \{0, \ldots, Q(n) - 1\}$, let a_{q+1} be any real number in (e_q, e_{q+1}). Denote by $A(n) = \{a_1, \ldots, a_{Q(n)}\}$ the collection of all such values.

Proposition 4.3. *The set of different scoring rules with n agents and 3 alternatives is $E(n) \cup A(n)$. Hence, the number of different scoring rules is $2|E(n)| - 1$.*

Clearly, by construction, every $e_q \in E(n)$ represents a particular scoring rule, different from all others. Furthermore, as shown in the proof of Proposition 4.3, every two values S_1 and S_2 in (e_q, e_{q+1}), $q \in \{0, \ldots, Q(n) - 1\}$, represent the same scoring rule, that is a different scoring rule to any other $e_q \in E(n)$, and also different from any other S in $(e_{q'}, e_{q'+1})$, $q' \in \{0, \ldots, Q(n) - 1\}$, with $q \neq q'$. As a result, the set of different scoring rules with n agents and 3 alternatives can be described as the solutions $E(n)$ plus any value in between two of these scoring rules. This makes a total number of $2|E(n)| - 1$ different scoring rules.

[15]We have programmed a Mathematica code that provides the set of solutions $S \in (0,1)$ to the system of inequalities (4.2). The code is available in the webpages above mentioned.

4.4. **Insights on Larger Societies.** We can offer a result on the best scoring rules that maximize the consistency index I for the maximax SWF. To this end, recall that e_1 denotes the minimum value of $E(n) \setminus \{0\}$. Note that in the 2×3 case, both plurality and the scoring rule with value in $(0, e_1)$ were the best for maximax. Proposition 4.4 shows that this result is robust to larger societies. Moreover, Proposition 4.4 shows that when n tends to infinity, plurality is *the* best scoring rule in terms of maximax.

Proposition 4.4. *In societies with n voters and 3 alternatives, plurality and the scoring rule with value in $(0, e_1)$ perform the best among the set of all scoring rules in terms of maximax. Moreover, when n tends to infinity, e_1 tends to zero.*

From the proof of Proposition 4.4, it is clear that these results are valid for any possible culture, provided that the utility realizations are i.i.d.

In the 2×3 maximin case, the best scoring rule turns out to be explosion, i.e., the one with the scoring value immediately below negative. Moreover, negative is strictly dominated by explosion. We now show that the latter result holds for any value of n. We also show that for larger societies, the optimal scoring rule in terms of maximin is never the interval of scoring rules immediately below negative.

Proposition 4.5. *For any n, negative is not the best scoring rule for maximin. Moreover, if $n \geq 7$, then the optimal scoring rule for maximin S^{MIN} is below $(n-1)/n$.*

The intuition behind the above result goes as follows. Consider a single realization u that is known to be the lowest utility value of an individual. Also, take a group of n realizations known to be middle utility values of n individuals. If n is small, the lowest realization in this group is likely to be above u. This is in fact the case in Theorem 3.1, for type 4 societies. However, as n increases, there is a competitive effect among the middle alternatives that takes the lowest of such realizations to be below u. We conjecture, however, that this competitive effect is reduced significantly if one adds additional worst realizations to both sides of the above argument. In fact, the simulations below unambiguously indicate this to be the case, making the optimal set of scoring rules converge to negative.

In the 2×3 utilitarian case, we observe that the sum of two middle realizations is close to the sum of a worst and a best realization (.35 and .3 in terms of the consistency index I), that nevertheless do not coincide. This is due to the dependence relation between the ordered utility values of the agents. We conjecture that as n increases, the dependence relations that explain this small deviation cancel out, making Borda the optimal scoring rule in the limit. Although we do not have a proof for this, the simulations below suggest that this is the case.

4.5. **Simulations.** In this section we construct a simulation model to study larger societies. Specifically, we analyze for $n = 2, 3, ..., 10$ and 100 the performance of all different scoring rules (see section 4.3) in terms of the three most prominent ideals of justice: utilitarianism, maximin, and maximax, under the ICC.

We proceed as follows. First, we obtain m different societies, for each of which we compare the alternatives selected by the three SWFs with those selected by each of the scoring rules. For a SWF W and a scoring rule S, we define $\tilde{x}_S$ as the random variable representing the number of times, out of the m profiles, that the scoring rule succeeds in selecting the alternative chosen by the SWF. We estimate $I(C, W, S)$ by constructing a confidence interval around the sample consistency index $\tilde{I}_S = \frac{\tilde{x}_S}{m}$.

Notice that $\tilde{x}_S$ follows a binomial distribution with parameters m (number of trials) and I (probability of success of a single trial). Hence, its expectation is given by $E[\tilde{x}_S] = m \cdot I$ and its variance is given by $V[\tilde{x}_S] = m \cdot I(1-I)$. For m large enough, we approximate the binomial distribution by using a normal distribution with the same expectation and variance. Then, the length of the confidence interval for I_S is given by

$$2z_{\alpha/2} \cdot \sqrt{\frac{I(1-I)}{m}} \tag{4.3}$$

Now, the greatest possible value for $I(1-I)$ is .25, which is reached when $I = .5$. Since the true value of I is unknown, taking the most conservative estimate, we consider the length of the interval as if $I(1-I) = .25$. We adopt a level of confidence of 95%, and an interval of length .001. Then, from (4.3) we obtain that $m = 3,841,600$. Hence, we simulate almost four million societies.

4.5.1. *Results of the simulation model.* We say that a scoring rule is *optimal* in terms of an ideal of justice if it cannot be statistically rejected that its consistency index is among the highest. Figure 2 reports the corresponding intervals of optimal scoring rules for the cases of utilitarianism and maximin, by providing the lower and upper values of S in that interval. Further, for each case Figure 2 also reports the scoring rule with the highest consistency index. We refer to the latter by the *best* scoring rules. Let us highlight the following results.

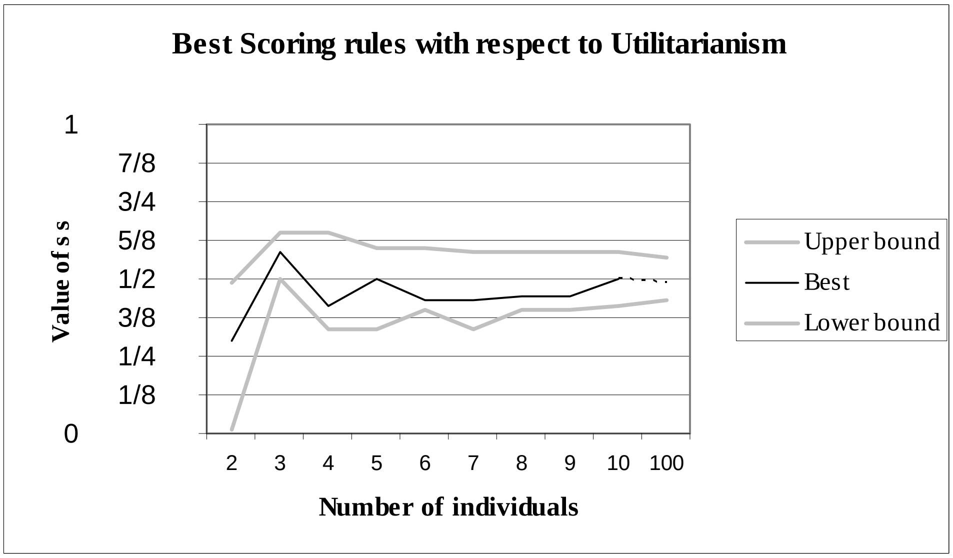


FIGURE 2. Results of the simulations for Utilitarianism and Maximin

First, note that for every n under consideration, the range of optimal scoring rules for maximin is in the set of explosion rules, while for utilitarianism it is generally (except for $n = 3$) in the range of implosion rules.[16]

[16]Having characterized the maximax case in Proposition 4.4, we omit the corresponding simulation results. It is sufficient to mention here that they are fully in line with what was theoretically obtained. We can provide the results upon request.

Second, as the size of the society increases, the set of optimal scoring rules for utilitarianism converges to an implosion rule close to Borda ($S = 1/2$). In terms of maximin, the set of optimal scoring rules converges to a high value explosion rule, close to negative. However, negative ($S = 1$) does not belong to the set of optimal scoring rules in terms of maximin. This is in concordance with what was obtained in Proposition 4.4.

We may conclude from the above that the theoretical picture obtained for the 2×3 case is closely mirrored in larger societies.

We now turn our attention to the values of the consistency index both for the best scoring rules and for plurality, Borda, and negative. Figure 3 reports these values. It can be immediately perceived that while the value of the consistency index of the best scoring rules is fairly stable around the level of 75% in terms of utilitarianism, it decreases considerably in terms of the maximin principle. That is, as n increases, the probability that the optimal scoring rules will select the utilitarian alternative is broadly stable, while it becomes harder and harder for them to select the maximin alternative (the maximax case is analogous to maximin). This may plausibly be due to the symmetry found in utilitarianism. It is the only SWF among those studied here that weights equally the utilities of both individuals, irrespective of their social position (worst-off or best-off). In the case of maximin (and maximax) the utility of only one of these individuals is considered. However, any scoring rule assigns the same weight to all individuals thus making such discrimination impossible.

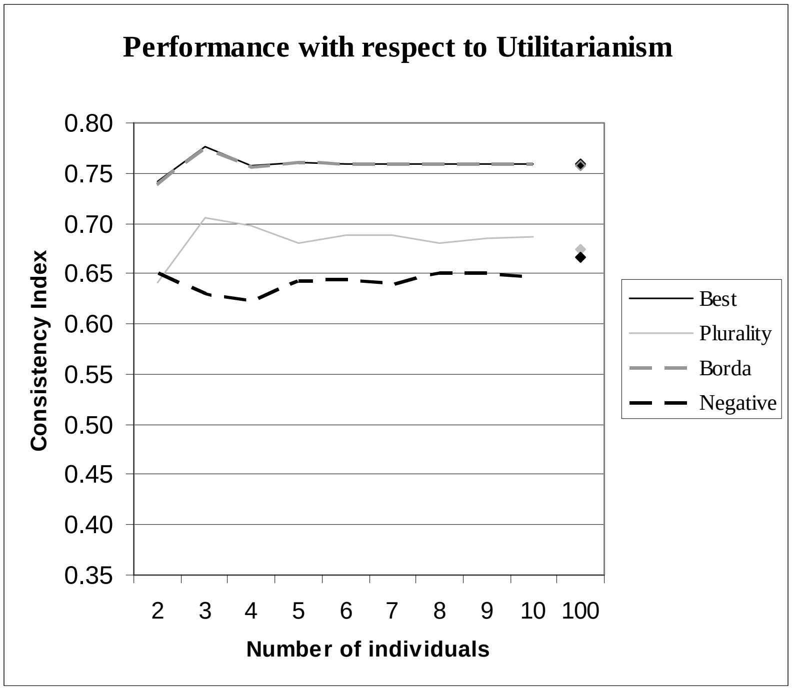
Second, Figure 3 shows that the differences between the best utilitarian scoring rule and the Borda count, and the best maximin scoring rule and negative are very small. However, as noted earlier, these differences are statistically significant.

Finally, the figure confirms the impression gained in the analysis of the 2×3 case regarding the poor performance of plurality in terms of utilitarianism and maximin.

5. Conclusions

This paper explores the relation between ideals of justice and voting systems. Whereas ideals of justice are typically presented in cardinal terms, the theory of voting systems is primarily constructed on the basis of ordinal information. We explore the cardinal consequences of using ordinal-based scoring rules.

We have investigated how the ordinal-cardinal relation evolves when different criteria of distributive justice are considered. We have shown that the optimal choice of scoring rule depends on the criterion of justice that one



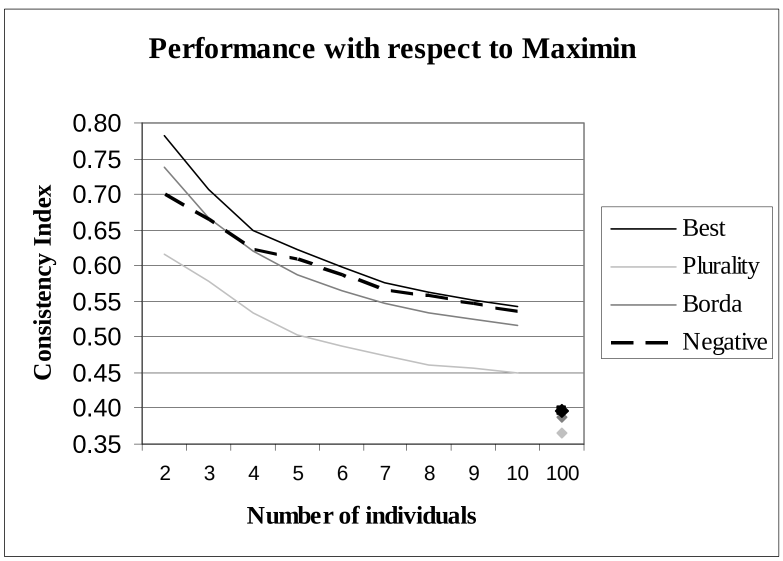


FIGURE 3. Performance of the main scoring rules

wishes to follow. Among our specific findings we emphasize that our results reinforce the natural link between utilitarianism and the Borda count (more specifically, between utilitarianism and a set of scoring rules converging to the Borda count). Also highlighted herein is the poor performance of plurality voting in terms of utilitarianism and maximin, as opposed to maximax where it performs optimally. Finally, we note that negative voting best approaches maximin.

Appendix A. Proofs

Proof of Theorem 3.1. We organize the proof by types of societies in ordinal terms U_O. The analysis of the first three types of societies is independent of the SWF under consideration.

Type 1: Perfect correlation. This is the case when $\succeq_1 = \succeq_2$. It is easy to see that this type has a measure of 1/6. The three SWFs always select the alternative regarded as the best element by both individuals, as any scoring rule except for negative voting. Negative randomizes between the best and the middle alternatives with equal probability. Therefore, for type 1 societies there is perfect consistency between the three SWFs and all scoring rules except negative voting. In the latter there is a consistency of 1/2.

Type 2: Only best-correlation. Both individuals agree on the best alternative, but disagree on the others (measure 1/6). All scoring rules coincide with the three SWFs in selecting the best alternative.

Type 3: Only worst-correlation. Both individuals coincide in the worst alternative, but disagree on the others (measure 1/6). All scoring rules attach equal selection probabilities to the non-worst alternatives. By the impartial culture condition, ICC, it is straightforward to see that in one half of the societies any of the three SWFs selects one of these non-worst alternatives, while in the other half the other non-worst alternative. Hence, all scoring rules will coincide with any of the three SWFs one half of the time.

Now we turn to analyze the case of type 4 and type 5 societies independently, for each of the three SWFs.

(i) **Utilitarianism**:

Type 4: Only middle-correlation. Both individuals agree on the middle alternative, but disagree on the others (measure 1/6). For every possible society: (i) plurality and implosion will randomize with equal probability between the non-middle alternatives, (2) Borda will randomize with probability 1/3 between the three alternatives, and (3) negative and explosion will select the middle alternative.

To compute the selection of utilitarianism we fist need to derive the joint probability distribution associated to the continuous random variables $Best_i$, $Middle_i$ and $Worst_i$, with $1 \geq Best_i \geq Middle_i \geq Worst_i \geq 0$, for a given individual i:

$$F(b_i, m_i, w_i) = P(Best_i \leq b_i, Middle_i \leq m_i, Worst_i \leq w_i) =$$

$$\begin{aligned} P(u_{i1} &\leq w_i, u_{i2} \leq w_i, u_{i3} \leq w_i) + \\ 3P(w_i &\leq u_{i1} \leq m_i, u_{i2} \leq w_i, u_{i3} \leq w_i) + \\ 3P(w_i &\leq u_{i1} \leq m_i, w_i \leq u_{i2} \leq m_i, u_{i3} \leq w_i) + \\ 6P(m_i &\leq u_{i1} \leq b_i, w_i \leq u_{i2} \leq m_i, u_{i3} \leq w_i) = \\ & w_i^3 + 3(m_i - w_i)w_i^2 + 3(m_i - w_i)^2 w_i + 6(b_i - m_i)(m_i - w_i)w_i \end{aligned}$$

The joint probability density function is

$$f(b_i, m_i, w_i) = \begin{cases} 6 \text{ if } 1 \geq b_i \geq m_i \geq w_i \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

Hence, the density function for the two agents is

$$f(b_i, m_i, w_i, b_j, m_j, w_j) = \begin{cases} 36 \text{ if } 1 \geq b_i \geq m_i \geq w_i \geq 0, \text{ and } 1 \geq b_j \geq m_j \geq w_j \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

We can now calculate the probability of utilitarianism choosing the middle alternative. The middle alternative will be the winner of utilitarianism, whenever the sum of m_i and m_j is greater than $b_i + w_j$ and $b_j + w_i$. Or equivalently,

$$\text{(A.1)} \qquad \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H_a}} \int_{\mathbf{H_b}} f(b_i, m_i, w_i, b_j, m_j, w_j) db_i db_j dm_i dm_j dw_i dw_j$$

where H_a and H_b are the sets $H_a = \{t : t \leq m_i + m_j - w_i\}$ and $H_b = \{t : t \leq m_i + m_j - w_j\}$. Then, (A.1) can be expressed as

(A.2)

$$\int_0^1 \int_0^1 \int_{w_j}^1 \int_{w_i}^1 \int_{m_j}^{\min\{1, m_i + m_j - w_i\}} \int_{m_i}^{\min\{1, m_i + m_j - w_j\}} 36 db_i db_j dm_i dm_j dw_i dw_j$$

which can be developed[17] as follows:

$$36 \Bigg[\int_0^1 \int_0^{w_j} \int_{w_j}^1 \Bigg(\int_{w_i}^{1 - m_j + w_i} (m_j - w_j)(m_i - w_i) dm_i +$$

[17] Some of the integrals are developed throughout the proof, especially in the first steps. These decompositions can be used later with any standard computer software to check results.

$$+\int_{1-m_j+w_i}^{1+w_j-m_j}(m_j-w_j)(1-m_j)dm_i+$$

$$+\int_{1+w_j-m_j}^{1}(1-m_i)(1-m_j)dm_i\Bigg)dm_jdw_idw_j+$$

$$+\int_0^1\int_{w_j}^1\int_{w_i}^1\Bigg(\int_{w_j}^{1-m_i+w_j}(m_j-w_j)(m_i-w_i)dm_j+$$

$$+\int_{1-m_i+w_j}^{1+w_i-m_i}(m_i-w_i)(1-m_i)dm_j+$$

$$+\int_{1+w_i-m_i}^{1}(1-m_i)(1-m_j)dm_j\Bigg)dm_idw_idw_j\Bigg]=.3$$

Clearly, utilitarianism chooses either of the two non-middle alternatives with equal probability, which, given the above result, has a value of .35. Then, plurality and implosion select the utilitarian selection with probability .35, Borda with probability 1/3, and explosion and negative with .3.

Type 5. No correlation. Individuals do not agree in the position of any alternative (measure 1/3). Note that then there exists an alternative (without loss of generality denoted by 1) that is evaluated by individuals i and j as best and middle, and another alternative (w.l.o.g. denoted by 2) evaluated as middle and worst. Hence, alternative 1 dominates alternative 2, and consequently utilitarianism will never select alternative 2. Note then that alternative 3 is evaluated as worst and best by individuals i and j. Then, the probability of utilitarianism choosing alternative 3 is

$$\text{(A.3)}\qquad \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_c}} f(b_i,m_i,w_i,b_j,m_j,w_j)db_idm_idw_idb_jdm_jdw_j$$

where H_c is $H_c=\{t:t\leq w_i+b_j-m_j\}$. This can be expressed as:

$$\text{(A.4)}\qquad \int_0^1\int_{w_j}^1\int_{m_j}^1\int_0^1\int_{w_i}^1\int_{\min\{w_i+b_j-m_j,m_i\}}^{\min\{1,w_i+b_j-m_j\}}36db_idm_idw_idb_jdm_jdw_j$$

which can be developed as follows:

$$36\int_0^1\int_{w_j}^1\int_{m_j}^1\Bigg(\int_0^{1+m_j-b_j}(1/2)(b_j-m_j)^2dw_i+$$

$$+\int_{1+m_j-b_j}^{1}(1/2)(1-w_i)^2dw_i\Bigg)db_jdm_jdw_j=1/5$$

Then, alternative 1 is selected by utilitarianism with probability 4/5. Thus, all scoring rules, except plurality, select the utilitarian winner with probability 4/5, and plurality with probability 1/2.

Table 1 summarizes the results with regard to utilitarianism, which com-

TABLE 1. Consistency Indexes for Utilitarianism

	Measure	Plurality	Implosion	Borda	Explosion	Negative
Type 1	1/6	1	1	1	1	1/2
Type 2	1/6	1	1	1	1	1
Type 3	1/6	1/2	1/2	1/2	1/2	1/2
Type 4	1/6	.35	.35	1/3	.3	.3
Type 5	1/3	1/2	4/5	4/5	4/5	4/5
Index		.642	.742	.739	.733	.65

pletes the proof of the first part of Theorem 3.1.

(ii) **Maximin**:

Type 4: Only middle-correlation. Let us calculate the probability of maximin choosing one of the non-middle alternatives. This will be the case when the middle value for player i is lower than the worst value for player j, $i \neq j$. Then, the probability of maximin selecting one of the non-middle alternatives (the worst for individual j) is

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_d}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i \tag{A.5}$$

where H_d is the set $H_d = \{t : t \geq m_i\}$ and $\mathbb{R}$ denotes the reals.[18] The previous probability is equal to 1/5. By symmetry, the probability of maximin choosing the other non-middle alternative is also 1/5, and hence the probability of choosing the middle alternative is 3/5.

Hence, plurality and implosion select the maximin selection with probability 1/5, Borda with probability 1/3, and explosion and negative with 3/5.

Type 5. No correlation. Taking the same notation as for the case of utilitarianism, the same argument as above (equation (A.5)) shows that the probability of maximin selecting alternative 3 is 1/5, and consequently the probability of selecting alternative 1 is 4/5.

[18]Note that equation (A.5) can be expressed as follows:

$$\int_0^1\int_{w_i}^1\int_{m_i}^1\int_{\mathbf{m_i}}^1\int_{w_j}^1\int_{m_j}^1 36 db_j dm_j dw_j db_i dm_i dw_i$$

Therefore, all the rules except plurality select the maximin option with probability 4/5, and plurality selects it with probability 1/2.

Table 2 summarizes the results for maximin. Then, the proof of the second

TABLE 2. Consistency Indexes for Maximin

	Measure	Plurality	Implosion	Borda	Explosion	Negative
Type 1	1/6	1	1	1	1	1/2
Type 2	1/6	1	1	1	1	1
Type 3	1/6	1/2	1/2	1/2	1/2	1/2
Type 4	1/6	1/5	1/5	1/3	3/5	3/5
Type 5	1/3	1/2	4/5	4/5	4/5	4/5
Index		.617	.717	.739	.783	.7

part of Theorem 3.1 is now complete.

(iii) **Maximax**: In type 4 societies (middle correlation), the maximax principle never selects the alternative evaluated as middle by the agents. Then, by symmetry, maximax randomizes with equal probabilities between the two non-middle alternatives.

In type 5 societies (no correlation), it is easy to see that maximax never selects the alternative that is evaluated as middle and worst by agents i and j, respectively. Then, the non-dominated alternatives for maximax are alternative 1, which is evaluated as best by i and middle by j, and alternative 3, which is evaluated as worst by i and best by j. Note that it cannot simultaneously hold that (i)$j's$ middle option is placed higher than $i's$ best option, and (ii) $i's$ worst option is placed higher than $j's$ best option. Then, alternative 1 will be the selected alternative by maximax whenever the best of i is above the best of j, and vice versa. Finally, since $i's$ and $j's$ best options are independent continuous random variables, maximax randomizes with equal probabilities between alternatives 1 and 3.

Table 3 concludes the proof. □

Proof of Theorem 3.2. The proof of Theorem 3.2 follows from the proof of Theorem 3.1, the analysis of type 4 and type 5 societies performed in section 3, and the resolution of integrals (3.2) and (3.4) contained in the supplement to this paper.□

Proof of Theorem 3.3. Note that for every $\lambda \in [0, 1]$ the values of the consistency index of implosion, Borda, and explosion coincide in all types of societies except type 4, where, for every $\lambda \in [0, 1]$ Borda obtains a consistency

Table 3. Consistency Indexes for Maximax

	Measure	Plurality	Implosion	Borda	Explosion	Negative
Type 1	1/6	1	1	1	1	1/2
Type 2	1/6	1	1	1	1	1
Type 3	1/6	1/2	1/2	1/2	1/2	1/2
Type 4	1/6	1/2	1/2	1/3	0	0
Type 5	1/3	1/2	1/2	1/2	1/2	1/2
Index		.667	.667	.639	.583	.5

index of 1/3. This is the case because Borda gives the same score to the 3 alternatives. However, the values of the consistency index of explosion and implosion depend on λ. Now, we have shown that (3.1) is a monotone and continuous function, and hence there must exist at least one intersection point with 1/3. To obtain this intersection we have to find the roots of the cubic function

$$\frac{5\lambda + 3\cdot\lambda^2 - 2\cdot\lambda^3}{10} = \frac{1}{3} \tag{A.6}$$

In order to avoid complex numbers we use Chebyshev cube roots instead of ordinary cube roots. Then, we first make a simple transformation of (A.6) to get a representation of the form $\lambda^3 + a\lambda^2 + b\lambda + c = 0$:

$$\lambda^3 - \frac{3}{2}\lambda^2 - \frac{5}{2}\lambda + \frac{5}{3} = 0 \tag{A.7}$$

We now reduce (A.7) to the depressed form by substituting $\lambda = g - \frac{a}{3} = g - \frac{1}{2}$:

$$g^3 - \frac{13}{4}g + \frac{1}{6} = 0 \tag{A.8}$$

The Chebyshev cube root is

$$C(t) = \sqrt{3}\cos(\arcsin(\frac{t}{2})/3) + \sin(\arcsin(\frac{t}{2})/3) \tag{A.9}$$

The solutions to (A.8) are

$$r_1 = \sqrt{t}C(t^{-\frac{3}{2}}q),$$
$$r_2 = -\sqrt{t}C(-t^{-\frac{3}{2}}q),$$
$$r_3 = -r_1 - r_2,$$

where $t = (a^2 - 3b)/9$ and $q = -(2a^3 - 9ab + 27c)/27$. The only solution in the unit interval is r_3. Therefore, the intersection of (3.2) with 1/3 of interest to us is:

$$x = r_3 + \frac{1}{2} = \frac{1}{2} + \sqrt{\frac{13}{3}} \sin\left(\frac{1}{3} \arcsin\left(\frac{2\sqrt{\frac{3}{13}}}{13}\right)\right) \simeq .551 \tag{A.10}$$

□

Proof of Theorem 4.1. Let $g(v) = -(1-\gamma)v^2 + (1-\gamma)v + \gamma$, with $v \in \{b_i, m_i, w_i, b_j, m_j, w_j\}$. The γ-joint density function is

$$f^\gamma(b_i, m_i, w_i, b_j, m_j, w_j) = \begin{cases} 36 \cdot g(b_i)g(m_i)g(w_i)g(b_j)g(m_j)g(w_j) \\ \text{ if } 1 \geq b_i \geq m_i \geq w_i \geq 0, \text{ and } 1 \geq b_j \geq m_j \geq w_j \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

The proof follows by reproducing the proof of Theorem 3.1 with the γ-joint density function.□

Proof of Theorem 4.2. The proof follows the spirit of of Theorem 3.1 by again dividing all possible societies into the same five types. To convey the main intuition, we report here only the integrals for the case of maximin in type 5 societies with regard to negative, and leave the remaining cases to the supplement of this paper.

In type 5 societies, negative selects alternative 1 (b_i and m_j). The social value of alternative 1 is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H_e}} \int_{\mathbb{R}} b_i f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i$$

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H_f}} \int_{\mathbb{R}} m_j f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i$$

where H_e and H_f are the sets $H_e = \{t : t \leq b_i\}$ and $H_f = \{t : t \geq b_i\}$. Notice that sets H_e and H_f establish conditions for b_i and m_j to be the maximin values of alternative 1. The above is equivalent to

$$36\Bigg(\int_0^1 \int_{w_i}^1 \int_{m_i}^1 \int_0^{b_i} \int_{b_i}^1 \int_{m_j}^1 b_i db_j dm_j dw_j db_i dm_i dw_i +$$

$$\int_0^1 \int_{w_i}^1 \int_{m_i}^1 \int_{b_i}^1 \int_{w_j}^1 \int_{m_j}^1 b_i db_j dm_j dw_j db_i dm_i dw_i +$$

$$\int_0^1 \int_{w_i}^1 \int_{m_i}^1 \int_0^{b_i} \int_{w_j}^{b_i} \int_{m_j}^1 m_j db_j dm_j dw_j db_i dm_i dw_i \Bigg) = \frac{13}{28}$$

The remaining types of societies can be calculated in a similar fashion, obtaining values $\frac{71}{140}$, $\frac{9}{14}$, $\frac{13}{27}$, and $\frac{13}{35}$ for types 1 to 4, respectively. Weighting these with the corresponding probability measures, negative obtains a social maximin value of $\frac{17}{35}$.

Independently of the type of society, the maximin value of the worst alternative is the lowest utility value among all realizations. Clearly this corresponds to $\min\{w_i, w_j\}$. Then, the expected value of this variable is

$$2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_g}}\int_{\mathbb{R}}\int_{\mathbb{R}} w_j f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i = \frac{1}{7}$$

where $H_g = \{t : t \leq w_i\}$.

To obtain the social value of the best alternative for maximin in type 5 societies, first note that alternative 2 is dominated. The social value can be b_i, m_j, w_i, or b_j:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_h}}\int_{\mathbb{R}} b_i f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i +$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_i}}\int_{\mathbb{R}} m_j f(b_i, m_i, w_i, b_j, m_j, w_j) db_j dm_j dw_j db_i dm_i dw_i +$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_j}}\int_{\mathbb{R}}\int_{\mathbb{R}} w_i f(b_i, m_i, w_i, b_j, m_j, w_j) db_i dm_i dw_i db_j dm_j dw_j +$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbf{H_k}}\int_{\mathbb{R}}\int_{\mathbb{R}} b_j f(b_i, m_i, w_i, b_j, m_j, w_j) db_i dm_i dw_i db_j dm_j dw_j$$

with $H_h = \{t : t \geq b_i\}$, $H_i = \{t : w_i \leq t \leq b_i\}$, $H_j = \{t : m_j \leq t \leq b_j\}$, $H_k = \{t : t \geq b_j\}$. The calculus of these integrals leads to a value of $\frac{69}{140}$. Weighting this value and those corresponding to the remaining types of societies, one can see that the election of the best maximin alternative obtains a value of $\frac{19}{35}$. Finally, the value of E for negative with respect to maximin is $\frac{\frac{17}{35}-\frac{1}{7}}{\frac{19}{35}-\frac{1}{7}} = \frac{6}{7}$. □

Proof of Proposition 4.3. That elements in $E(n)$ define different scoring rules is obvious from the definition of $E(n)$. We now claim that, for any two scoring rules S_s and S_t with $e_q < S_s < S_t < e_{q+1}$, $q \in \{0, \ldots, Q(n)-1\}$, both scoring rules are equivalent.

Suppose this is not true. In this case, there exists a profile of preferences in U_O such that the scoring rule associated to S_s selects, w.l.o.g., alternative 1 whereas 2 is among those selected by S_t (where either 1 is not selected by S_t or 2 is not selected by S_s. We assume w.l.o.g. that 1 is not selected by S_t). Then $B^1 + S_s M^1 \geq B^2 + S_s M^2$ (with $B^1 + S_s M^1 \geq B^3 + S_s M^3$) and

$B^1 + S_t M^1 < B^2 + S_t M^2$ (with $B^3 + S_t M^3 \leq B^2 + S_t M^2$). Equivalently, $(B^1 - B^2) \geq S_s(M^1 - M^2)$ and $(B^1 - B^2) < S_t(M^1 - M^2)$. Since $(B^1 - B^2) < S_t(M^1 - M^2)$, it must be either $B^1 \neq B^2$ or $M^1 \neq M^2$. Thus, if $(B^1 - B^2) = S_s(M^1 - M^2)$, it should be S_s in $E(n)$, which is absurd. Therefore, it must be $(B^1 - B^2) > S_s(M^1 - M^2)$. Thus, there exists a value S_g in the interval (S_s, S_t) such that $(B^1 - B^2) = S_g(M^1 - M^2)$ which leads to $S_g \in E(n)$ again contradicting the construction of set $E(n)$.

Moreover, if $S \in (e_q, e_{q+1})$, it is clear that no ties (with different values B^l, M^l) can appear for the associated scoring rule, thus making it different from any scoring rule in $E(n)$.[19] As a result, the set of scoring rules with n agents can be described as the solutions $E(n)$ plus any value in between two of these scoring rules.□

Proof of Proposition 4.4. First note that only alternatives regarded as best by some agents can be selected by maximax. In fact, due to the i.i.d. nature of utility realizations across individuals, those alternatives that the larger number of agents regard as best are the ones with the highest probability of being the maximax alternative. It is immediate that plurality selects precisely this type of alternatives (those with the larger number of votes) as best. Now, consider e_1 as defined above, and take any scoring rule S with $S \geq e_1$. By construction, it is clear that there exists a profile of preferences in U_O for which the alternative with the larger number of votes as best alternative is displaced by (or selected in combination with) a different one with a strictly smaller number of votes. By our previous reasoning, the choice made by such a scoring rule leads to a lower consistency index than plurality.

To study the second part of the Proposition, we prove that for any m in the natural numbers, there exists a society for which all scoring rules with $S \geq \frac{1}{m}$ are worse than plurality in terms of maximax. Consider a society with dimension $n = 2m + 1$ and let $B = (m + 1, m), M = (0, m)$. Note that, for any scoring rule, the third alternative is strictly dominated by alternative 2. Any scoring rule with $S \geq \frac{1}{m}$ considers alternative 2 among the set of selected alternatives, thus underperforming plurality. Hence, the set of best scoring rules in terms of maximax when $n = 2m + 1$ is contained in the set of scoring rules with value in $[0, \frac{1}{m})$.□

Proof of Proposition 4.5. Recall that $e_{Q(n)-2}$ and $e_{Q(n)-1}$ denote the two largest values that solve the system 4.2, that correspond to $(n-2)/(n-1)$ and $(n-1)/n$, respectively. Now consider the following two types of ordinal societies: (1) society-1 (perfect correlation): this is the case when $\succeq_i = \succeq_j$ for

[19] It is straightforward to see that they are different from the extremes 0 and 1.

every $i, j \leq n$; and (2) society-2: all individuals except one have the same ordering, while the other individual has the reverse ordering. W.l.o.g., let the individual with a different ordering be individual n, and to avoid confusion with future developments, denote this individual by the symbol $\diamond$.

We show that among the four following scoring rules, $S_a \in ((n-2)/(n-1), (n-1)/n)$, $S_b = (n-1)/n$, $S_c \in ((n-1)/n, 1)$, and $S_d = 1$, S_a dominates the other scoring rules in the sense of the consistency index I. First note that the four scoring rules select the same alternative in all ordinal societies, except in the two above. Thus, they have the same probability of choosing the maximin winner in every society, except for the two above.

Consider society-1 (perfect correlation). Clearly, S_a, S_b and S_c always select the alternative regarded as best by all agents, while S_d randomizes between this alternative and the one regarded as the middle option by all agents. Hence, since this society has a positive measure ($1/6^n$) and maximin always selects the alternative regarded as best by all agents, S_d is dominated for this type of ordinal societies.

Now consider society-2. Denote by 1 the alternative that is regarded by all agents as the middle option, by 2 the alternative regarded by all individuals, except $\diamond$, as best, the remaining alternative is denoted by 3. We compute the probability of maximin selecting alternative 2. The condition for 2 not being the maximin winner is $u_{\diamond 2} \leq u_{i1}$, $i = 1, \ldots, n-1$. Note that $u_{\diamond 2} = w_\diamond$ and $u_{i1} = m_i$, $i = 1, \ldots, n-1$. The density function of the worst variable is $3(1-w_\diamond)^2$ and the density function of a middle variable is $6m_i(1-m_i)$. Therefore, the probability of maximin selecting alternative 2 is

$$\rho_2 = \int_0^1 \int_{w_\diamond}^1 \cdots^{n-1\ times} \cdots \int_{w_\diamond}^1 3(1-w_\diamond)^2 \prod_{i=1}^{n-1} 6m_i(1-m_i) dm_{n-1} \cdots dm_1 dw_\diamond.$$

This is equivalent to $\rho_2 = \int_0^1 3(1-w_\diamond)^2[1-(w_\diamond)^2(3-2w_\diamond)]^{n-1} dw_\diamond$.

We now calculate the probability of maximin choosing the third alternative, ρ_3. This will be the case whenever $w_i > m_\diamond$ for every $i \leq n-1$. Therefore,

$$\rho_3 = \int_0^1 \int_{m_\diamond}^1 \cdots^{n-1\ times} \cdots \int_{m_\diamond}^1 6m_\diamond(1-m_\diamond) \prod_{i=1}^{n-1} 3(1-w_i)^2 dw_{n-1} \cdots dw_1 dm_\diamond =$$

$$\int_0^1 6m_\diamond(1-m_\diamond)^{3n-2}$$

When $n = 6$, $(\rho_1, \rho_2, \rho_3) = (\frac{46}{91}, \frac{2204}{4641}, \frac{1}{51})$, and thus $\rho_1 > \rho_2 > \rho_3$. However, when $n = 7$, $(\rho_1, \rho_2, \rho_3) = (\frac{21627}{45220}, \frac{22947}{45220}, \frac{1}{70})$, hence $\rho_2 > \rho_1 > \rho_3$. Therefore, for this type of societies, when $n = 7$ the maximin winner is more likely to be alternative 2. Scoring rule S_a also selects alternative 2, while S_b randomizes with equal probabilities between alternatives 1 and 2, and S_c and S_d select alternative 1. Hence, since this society has a positive measure $(n/6^n)$, S_a dominates the other scoring rules for this type of ordinal societies.

Considering the analysis of both types of societies and taking into account that S_a,S_b,S_c and S_d always select the same alternative in any other type of society, it is clear that S_a has the largest index among this group of scoring rules for $n = 7$.

To conclude the proof of the proposition, it is sufficient to note that ρ_2 is increasing with respect to n. To see this, recall that the condition for 2 not being the maximin winner is $u_{\diamond 2} \leq u_{i1}$, $i = 1, \ldots, n-1$. Clearly, from the independence generation of utilities across individuals, the larger n the stricter the condition and thus, the probability of alternative 2 not being selected is smaller. This proves that ρ_2 is increasing with respect to n and since this probability is strictly larger than $\frac{1}{2}$ when $n = 7$, for any $n \geq 7$, it is $\rho_2 > \frac{1}{2} > \rho_1$, making S_a the best scoring rule among the four considered.□

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