A LOW DIMENSIONAL KALMAN FILTER FOR SYSTEMS WITH LAGGED OBSERVABLES

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Abstract. This note describes how the Kalman filter can be modified to allow for the vector of observables to be a function of lagged variables without increasing the dimension of the state vector in the filter. This is useful in applications where it is desirable to keep the dimension of the state vector low. The modified filter and accompanying code (which nests the standard filter) can be used to compute (i) the steady state Kalman filter (ii) the log likelihood of a parameterized state space model conditional on a history of observables (iii) a smoothed estimate of latent state variables and (iv) a draw from the distribution of latent states conditional on a history of observables.
Keywords: Kalman filter, lagged observables, Kalman smoother, simulation smoother

This note describes how the Kalman filter can be modified to allow for the vector of observable variables in the measurement equation to be a function of lagged variables. The standard approach, which is to augment the state vector of the filter to include also lagged variables, works fine in most applications. However, it also doubles the dimension of the state vector, which in some applications may be undesirable. The modified filter presented here avoids increasing the dimension of the state by exploiting that the innovation representation can be modified so as to make it unnecessary to augment the state vector with lagged variables. The derivation of the modified filter, which nests the standard filter as a special case, is presented in the next section.

As it turns out, the Kalman smoother and the Kalman simulation smoother can be computed using the standard backward recursions, once the modified filter has been used to compute the forward recursions. The reason is that there is no additional information in the observation vectors about the latent state that can be used in the backward recursions beyond what is already implied by the state estimate from the forward filter. Therefore, whether the vector of observables is a function of lagged variables or not, does not matter. To provide the reader with a Kalman "one stop shop", the formulas for the Kalman smoother and an algorithm for the Kalman simulation smoother are given in Section 3 and 4, though these sections contain nothing new beyond the original references, i.e. Hamilton (1994) and Durbin and Koopman (2002). Section 5 contains two examples, illustrating how a simple linearized general equilibrium model can be mapped into the state space system of the filtering problem. Section 6 concludes.

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1. A filtering problem

Consider the state space system

$$X_t = AX_{t-1} + Cu_t \tag{1.1}$$

$$Z_t = D_1 X_t + D_2 X_{t-1} + Ru_t \tag{1.2}$$

where X_t is the $n \times 1$ dimensional state vector, A is an $n \times n$ matrix, C is and $n \times m$ matrix and $u_t \sim N(0, N)$ is an $m \times 1$ vector of white noise Gaussian disturbances. Z_t is a $p \times 1$ vector of observable variables and D_1, D_2 and $p \times n$ matrices and R is a $p \times m$ matrix. Solving the filtering problem implies finding a recursive formula

$$X_{t|t} = AX_{t-1|t-1} + K_t \left(Z_t - AX_{t-1|t-1} \right) \tag{1.3}$$

for the linear minimum variance estimate of X_t conditional on the history of observations up to period t defined as

$$X_{t|t} = \arg\min_{\widehat{X}_t} E \left(X_t - \widehat{X}_t \right) \left(X_t - \widehat{X}_t \right)' \tag{1.4}$$

$$s.t. \quad X_{t|t} \equiv \sum_{s=0}^{t} \mathbf{a}_s Z_s - \mathbf{b} X_{0|0} \tag{1.5}$$

where $X_{0|0}$ is an initial estimate of the state X_0 with covariance $P_{0|0}$ and $X_{t|t-s} \equiv E\left(X_t | Z^{t-s}\right)$.

$$P_{0|0} \equiv E\left[\left(X_0 - X_{0|0} \right) \left(X_0 - X_{0|0} \right)' \right] \tag{1.6}$$

and $\mathbf{a}_s$ and $\mathbf{b}$ are vectors of appropriate dimension. The initial state estimation error is assumed to be orthogonal to the disturbances u_t, that is, $E\left[\left(X_0 - X_{0|0} \right) u'_{t+s} \right] = 0$ for all s. Solving the filtering problem implies finding K_t in (1.3) such that (1.4) - (1.5) holds.

The state space system (1.1) - (1.2) is standard apart from the fact that the vector of observables Z_t depends on the lagged state in addition to the current state. A straight forward and common way to get around this problem is to redefine the state so as to include also lagged X_t to get

$$\overline{X}_t = \overline{AX}_{t-1} + \overline{C} u_t \tag{1.7}$$

$$Z_t = \overline{DX}_t + Ru_t \tag{1.8}$$

where

$$\overline{X}_t = \begin{bmatrix} X'_t & X'_{t-1} \end{bmatrix}', \quad \overline{A} = \begin{bmatrix} A & \mathbf{0} \\ I & \mathbf{0} \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} C \\ \mathbf{0} \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$$

The standard filter can the be applied to the augmented system (1.7) - (1.8). In most application, this does not cause any complications. However, in some cases it is desirable to have a state of low dimension, and redefining the state as above causes a doubling of the dimension of the state, i.e. $\overline{X}_t$ is a $2n \times 1$ vector. Below a new filter is derived that avoids expanding the state and solves the filtering problem (1.3) - (1.5) while maintaining an n-dimensional state vector.

2. A modified filter

In this section, a modified filter is derived using the Gram-Schmidt approach of recursively orthogonalizing the observables (a derivation along similar lines of the standard filter can be found in Anderson and Moore (1979).) The basic idea is to exploit that the projection of a variable onto a set of orthogonal variables is equivalent to adding up the projections of the variable onto the individual variables. That is,

$$E(x \mid z, y) = E(x \mid z) + E(z \mid y) \tag{2.1}$$

if

$$E(zy') = 0 \tag{2.2}$$

and x, y and z are Gaussian random variables. Our strategy will be to add the projections of X_t onto period $t-1$ information and onto the component of period t information that is orthogonal to period $t-1$ information, that is $X_{t|t}$ will be computed as

$$X_{t|t} = X_{t|t-1} + E\left(X_t \mid \widetilde{Z}_t\right) \tag{2.3}$$

where

$$\widetilde{Z}_t \equiv Z_t - E\left(Z_t \mid X_{t-1|t-1}\right) \tag{2.4}$$

and $E(X_{t|t-1}\widetilde{Z}_t') = 0$. To find the recursive formula we thus need find an expression for $E\left(X_t \mid \widetilde{Z}_t\right)$.

2.1. The Initial Period.

There are two pieces of information available in period 1: The initial state estimate $X_{0|0}$ with covariance $P_{0|0}$ and the observation vector Z_1. The initial state estimate can together with (1.1) be used to construct a prior for X_1 as

$$X_{1|0} = AX_{0|0} \tag{2.5}$$

We now want to find a Kalman gain K_1 such that

$$X_{1|1} = AX_{0|0} + K_1\widetilde{Z}_1 \tag{2.6}$$

is the linear minimum variance estimate of X_1 conditional on Z_1 and $X_{0|0}$ and where

$$\widetilde{Z}_1 = Z_1 - E\left(Z_1 \mid X_{0|0}\right) \tag{2.7}$$

where by (1.1) and (1.2)

$$E\left(Z_1 \mid X_{0|0}\right) = D_1AX_{0|0} - D_2X_{0|0} \tag{2.8}$$

We can verify that $\widetilde{Z}_1$ is orthogonal to $X_{0|0}$ by writing the innovation $\widetilde{Z}_1$ as a function of the initial period error and the period 1 disturbance vector u_1

$$\begin{aligned} E\left(X_{0|0}\widetilde{Z}_1'\right) &= E\left(X_{0|0}\left(X_{0|0} - X_0\right)'(D_1A - D_2)' + X_{0|0}u_1'(D_1C + R)'\right) \\ &= 0 \end{aligned}$$

since $E\left(X_{0|0}\left(X_{0|0} - X_0\right)'\right) = E\left(X_{0|0}u_1'\right) = 0$ by assumption.

From the projection theorem (see Brockwell and Davis 2006) we know that K_1 is given by the projection formula

$$K_1 = E\left(X_1 \widetilde{Z}_1'\right)\left[E\left(\widetilde{Z}_1 \widetilde{Z}_1'\right)\right]^{-1} \tag{2.9}$$

We thus need to find $E\left(X_1 \widetilde{Z}_1'\right)$ and $E\left(\widetilde{Z}_1 \widetilde{Z}_1'\right)$.

2.2. **The covariance of the state and the innovation.** The covariance of the state and the innovation vector can be expanded by using the definition of the innovation (2.7)

$$E\left(X_1 \widetilde{Z}_1'\right) = \left[EX_1\left(Z_1 - D_1 A X_{0|0} - D_2 X_{0|0}\right)'\right] \tag{2.10}$$

and then further expanding by substituting out the observation Z_1 by using (1.1) - (1.2) to get

$$E\left(X_1 \widetilde{Z}_1'\right) = E[X_1\left(D_1\left(AX_0 + Cu_1\right) + D_2 X_0 + Ru_t - D_1 A X_{0|0} - D_2 X_{0|0}\right)'] \tag{2.11}$$

Now define the posterior state estimation error $\widetilde{X}_t$

$$\widetilde{X}_t = X_t - X_{t|t} \tag{2.12}$$

and use that to rearrange the expression for the covariance (2.11) so that

$$\begin{aligned} E\left(X_1 \widetilde{Z}_1'\right) &= E[\left(A\left(\widetilde{X}_0 + X_{0|0}\right) + Cu_1\right) \\ &\quad \times \left(\left(D_1 A + D_2\right)\widetilde{X}_0 + D_1 C u_1 + R u_1\right)'] \end{aligned} \tag{2.13}$$

Finally, the fact that $E\left(X_{0|0}\widetilde{X}_0'\right) = 0$ (since the period 0 state estimation error must be orthogonal to quantities known in period 0) allows us to simplify (2.13) to get

$$E\left(X_1 \widetilde{Z}_1'\right) = AP_{0|0}\left(D_1 A + D_2\right)' + CC'D_1' + CR'. \tag{2.14}$$

We thus have the first term in the Kalman gain (2.9).

2.3. **The covariance of the innovation vector.** Expand the second term in the Kalman gain (2.9) by again using the definition of the innovation (2.7) and the system definitions (1.1) - (1.2)

$$\begin{aligned} E\left(\widetilde{Z}_1 \widetilde{Z}_1'\right) &= \left(\left(D_1 A + D_2\right)\widetilde{X}_0 + \left(D_1 C + R\right) u_1\right) \\ &\quad \times \left(\left(D_1 A + D_2\right)\widetilde{X}_0 + \left(D_1 C + R\right) u_1\right)' \end{aligned} \tag{2.15}$$

and then use that the definition (1.6) of the period zero error covariance and the definition of $\widetilde{X}_0$ (2.12) together implies

$$P_{0|0} = E\left(\widetilde{X}_0 \widetilde{X}_0'\right) \tag{2.16}$$

to get

$$\begin{aligned} E\left(\widetilde{Z}_1 \widetilde{Z}_1'\right) &= \left(D_1 A + D_2\right) P_{0|0}\left(D_1 A + D_2\right)' \\ &\quad + \left(D_1 C + R\right)\left(D_1 C + R\right)' \end{aligned} \tag{2.17}$$

2.4. **The Kalman gain.** Plugging in (2.14) and (2.17) into the (2.18) then yields the Kalman gain for the first period

$$\begin{aligned} K_1 &= \left(AP_{0|0}\left(D_1A + D_2\right)' + CC'D_1' + CR'\right) \\ &\quad \times \left[\left(D_1A + D_3\right)P_{0|0}\left(D_1A + D_3\right)' + \left(D_1C + R\right)\left(D_1C + R\right)'\right]^{-1} \end{aligned} \tag{2.18}$$

2.5. **The posterior covariance.** To start the recursion and find a general expression for the period t Kalman gain K_t, we simply need to go through the same steps as above. However, in the initial period the covariance $P_{0|0}$ of the initial period estimate X_0 was given exogenously. To start the recursion we thus first need to find an expression for posterior covariance matrix in period 1, i.e $P_{1|1}$. This can be done as follows. We have

$$X_{1|1} = X_{1|0} + K_1\widetilde{Z}_1 \tag{2.19}$$

Add X_1 to each side and rearrange to get

$$X_1 - X_{1|1} + K_1\widetilde{Z}_1 = X_1 - X_{1|0} \tag{2.20}$$

Since the posterior error $X_1 - X_{1|1}$ must be orthogonal to the innovation $\widetilde{Z}_1$ the variance on the left hand side is just the sum of the variance of the error and the innovation, so that

$$P_{1|1} + K_1\left[\left(D_1A + D_2\right)P_{0|0}\left(D_1A + D_2\right)' + \left(D_1C + R\right)\left(D_1C + R\right)'\right]K_1' = P_{1|0} \tag{2.21}$$

rearranging gives

$$P_{1|1} = P_{1|0} - K_1\left[\left(D_1A + D_2\right)P_{0|0}\left(D_1A + D_2\right)' + \left(D_1C + R\right)\left(D_1C + R\right)'\right]K_1' \tag{2.22}$$

Since

$$X_{2|1} - X_2 = A\left(X_{1|1} - X_1\right) + Cu_t \tag{2.23}$$

the prior covariance in period 2 is given by

$$P_{2|1} = AP_{1|1}A' + CC' \tag{2.24}$$

2.6. **The Kalman recursions.** We now have all the ingredients we need for the Kalman recursions. Applying the same step (2.5) - (2.18) as in the initial period but for period t gives

$$\begin{aligned} K_t &= P_{t|t-1}D_1' && (2.25)\\ &\quad \times \left[\left(D_1A + D_2\right)P_{t-1|t-1}\left(D_1A + D_2\right)' + \left(D_1C + R\right)\left(D_1C + R\right)'\right]^{-1} \\ P_{t|t} &= P_{t|t-1} && (2.26)\\ &\quad -K_t\left[\left(D_1A + D_2\right)P_{t-1|t-1}\left(D_1A + D_2\right)' + \left(D_1C + R\right)\left(D_1C + R\right)'\right]K_t' \\ P_{t+1|t} &= AP_{t|t}A' + CC' && (2.27) \end{aligned}$$

The steady state Kalman gain K_∞ can as usual be found by iterating on (2.26) - (2.27) until convergence.

2.7. **Computing the log likelihood.** The fact that the innovations $\widetilde{Z}_t$ are i.i.d. Gaussian vectors can be used (just as with the standard filter) to recursively compute the log likelihood $\mathcal{L}$ of the data conditional on a parameterized state space system with Gaussian disturbances. It is given by

$$\mathcal{L}\left(Z \mid A, C, D1, D2, R\right) = -\frac{1}{2}\sum_{t=1}^{T}\left(p\ln\pi + \ln\left|\Omega_t\right| + \widetilde{Z}_t'\Omega_t^{-1}\widetilde{Z}_t\right) \tag{2.28}$$

where

$$\Omega_t \equiv E\left(\widetilde{Z}_t\widetilde{Z}_t'\right) \tag{2.29}$$

$$= \left(D_1A + D_2\right)P_{t-1|t-1}\left(D_1A + D_2\right)' + \left(D_1C + R\right)\left(D_1C + R\right)' \tag{2.30}$$

3. The Kalman smoother for the modified system

The smoothed estimate of X_t is defined as the linear minimum variance estimate of X_t conditional on the complete history of observables, i.e.

$$X_{t|T} = \arg\min_{\widehat{X}_t} E\left(X_t - \widehat{X}_t\right)\left(X_t - \widehat{X}_t\right)' \tag{3.1}$$

$$s.t. \quad X_{t|T} \equiv \sum_{s=0}^{T}\mathbf{a}_sZ_s - \mathbf{b}X_{0|0} \tag{3.2}$$

As shown in Hamilton (1994), there is no additional information in Z_t about the state X_t beyond what is already incorporated in the estimate $X_{t|t}$. This makes it possible to derive the smoother without an explicit role for the history of observables, once we have the recursions (2.25)-(2.27). The smoothed estimates of X_t are given by

$$X_{t|T} = X_{t|t} + J_{t-1}\left(X_{t+1|T} - X_{t+1|t}\right) \tag{3.3}$$

where

$$J_t = P_{t|t}A'P_{t+1|t}^{-1} \tag{3.4}$$

The covariances of the smoothed state estimation errors can be computed as

$$P_{t|T} = P_{t|t} + J_t\left(P_{t+1|T} - P_{t+1|t}\right)J_t'$$

(for more details, see Hamilton 1994).

3.1. A Kalman smoother algorithm.

(1) Compute the sequence $X_{t|t} : t = 1, 2, .., T$ using the forward recursions (1.3) and (2.25) - (2.27). Store $X_{t|t}, P_{t|t}$ and $P_{t+1|t}$.
(2) Compute the smoothed estimates $X_{t|T} : t = T-1, T-2, ..., 1$ using the backward recursions (3.3) - (3.4).

4. The Kalman simulation smoother for the modified system

As described in Durbin and Koopman (2002), a draw from $p(X^T|Z^T)$ can be generated by the following algorithm

4.1. **A Kalman simulation smoother algorithm.**

(1) Construct a draw Z^{+T} from $p(Z^T)$ using the system (1.1) (1.2) and save the draw of the state X^{+T}.
(2) Construct $Z^{*T} = Z^T - Z^{+T}$.
(3) $\widetilde{X}^T = \widehat{X}^{*T} + X^{+T}$ is then a draw from $p(X^T|Z^T)$ where $\widehat{X}^{*T} = E\left(X|^T Z^{*T}\right)$ (i.e. $\widehat{X}^{*T}$ is the output of running Z^{*T} through the smoothing algorithm above).

This algorithm has the advantage over some other simulation smoothers in that it only involves drawing from the i.i.d. vectors of u_t rather than from conditional distributions of the state x_t (with the exception of generating the draw from the distribution of the initial state $p(X_0)$). The latter is often singular in interesting economic applications, due to that the state dimension is often larger than the stochastic dimension in models with endogenous state variables. A singular covariance matrix requires additional computational steps which are avoided in Durbin and Koopman's algorithm.

5. Examples

Consider the simple New Keynesian model

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(y_t - \overline{y}_t) \tag{5.1}$$
$$y_t = E_t \pi_{t+1} - \sigma\left(i_t - E_t \pi_{t+1}\right) \tag{5.2}$$
$$i_t = \phi \pi_t \tag{5.3}$$
$$\overline{y}_t = \rho \overline{y}_{t-1} + \varepsilon_t \tag{5.4}$$

where $\pi_t, y_t, \overline{y}_{tt}, i_t$ are inflation, output, potential output and nominal interest rate respectively. This model has a single variable, potential output $\overline{y}_t$, as the state. The model can be solved to get

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = G\overline{y}_t \tag{5.5}$$

where

$$G = \begin{bmatrix} -\kappa \frac{\rho-1}{\rho+\beta\rho-\beta\rho^2-\kappa\sigma\phi+\kappa\sigma\rho-1} \\ -\kappa \frac{\sigma\phi-\sigma\rho}{\rho+\beta\rho-\beta\rho^2-\kappa\sigma\phi+\kappa\sigma\rho-1} \end{bmatrix} \tag{5.6}$$

5.1. **Two vintages of data.**

Assume that we have two vintages of noise ridden data so that in period t we can observe

$$Z_t = \begin{bmatrix} \pi_t \\ y_t \\ \pi_{t-1} \\ y_{t-1} \end{bmatrix} + \mathbf{v}_t \tag{5.7}$$

where v_t is a vector of measurement errors. We then have

$$\begin{aligned} X_t &= \overline{y}_t,\ A = \rho \\ C &= \left[\begin{array}{cc} \sqrt{E(\varepsilon_t)^2} & \mathbf{0}_{1\times p} \end{array} \right], \\ D1 &= \left[\begin{array}{c} G \\ \mathbf{0}_{2\times 1} \end{array} \right],\quad D2 = \left[\begin{array}{c} \mathbf{0}_{2\times 1} \\ G \end{array} \right] \\ R &= \left[\begin{array}{cc} \mathbf{0}_{4\times 1} & \left[E(\mathbf{v}_t\mathbf{v}_t')\right]^{1/2} \end{array} \right] \end{aligned}$$

5.2. **Data in 1st differences.**

$$Z_t = \left[\begin{array}{c} \Delta\pi_t \\ \Delta y_t \end{array} \right] + \widetilde{\mathbf{v}}_t \tag{5.8}$$

we then have

$$\begin{aligned} D1 &= G,\quad D2 = -G \\ R &= \left[\begin{array}{cc} \mathbf{0}_{2\times 1} & \left[E(\widetilde{\mathbf{v}}_t\widetilde{\mathbf{v}}_t')\right]^{1/2} \end{array} \right] \end{aligned}$$

6. Summing up

Above it was demonstrated how the Kalman filter can be modified to allow for lagged observables without increasing the dimension of the state vector in the filter. While the standard approach of augmenting the state vector with lagged variables works well in many applications, it also introduces additional computational burdens that in some applications may have non-negligible costs. The aim of this note is to provide a single reference for filtering problems with lagged observables in settings where it is desirable to keep the dimension of the state low, either to improve tractability of a model, increase computational speed or to reduce storage requirements.[1]

The accompanying code for the Kalman filter, smoother and simulation smoother is described in the Appendix.

References

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[3] Durbin, J. and S.J. Koopman, 2002,"A simple and efficient simulation smoother for state space time series analysis", Biometrica, 89, pp. 603-615.
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[5] Nimark, K, 2008, "Dynamic Pricing and Imperfect Common Knowledge", *Journal of Monetary Economics*, 55, pp. 365-382.

[1]The author has found that in applications to imperfect information models (which often have high dimensional representations, e.g. Nimark 2008) the modified filter improves tractability and computational performance.

Appendix A. Guide to MatLab code

M-files available at www.kris-nimark.net

A.1. **The Steady State filter.** The m-file Steady.m takes the matrices $A, C, D1, D2, R$ as inputs and returns the steady state Kalman gain K_t and $P_{t+1|t}$ for $t = \infty$.

$$[K, P] = steady(A, C, D1, D2, R)$$

A.2. **The Log-Likelihood.** The m-file `logl.m` takes the matrices $A, C, D1, D2, R$ and the data matrix $Z \equiv \left[\begin{array}{cccc} Z_1 & Z_2 & \cdots & Z_T \end{array} \right]$ as inputs and returns the log-likelihood $LL = \mathcal{L}\left(Z \mid A, C, D1, D2, R\right)$.

$$[LL] = \log l(A, C, D1, D2, R, Z)$$

A.3. **The Kalman Smoother.** The m-file `smooth.m` takes the matrices $A, C, D1, D2, R$ and the data matrix $Z \equiv \left[\begin{array}{cccc} Z_1 & Z_2 & \cdots & Z_T \end{array} \right]$ as inputs and returns the smoothed estimate $X = \left[\begin{array}{cccc} X_{1|T} & X_{2|T} & \cdots & X_{T|T} \end{array} \right]$

$$[X] = smooth(A, C, D1, D2, R, Z)$$

A.4. **The Kalman Simulation Smoother.** The m-file `sim.m` takes the matrices $A, C, D1, D2, R$ and the data matrix $Z \equiv \left[\begin{array}{cccc} Z_1 & Z_2 & \cdots & Z_T \end{array} \right]$ as inputs and returns a draw X from $p\left(X^T \mid Z^t\right)$.

$$[X] = sim(A, C, D1, D2, R, Z)$$

A.5. **The State Distribution Plotter.** The m-file `plotdist.m` takes the matrices $A, C, D1, D2, R$ and the data matrix $Z \equiv \left[\begin{array}{cccc} Z_1 & Z_2 & \cdots & Z_T \end{array} \right]$ as inputs together with the percentiles (`upper` and `lower`), the number of draws `ndraws`, and two indicators `plotplease` and `legendplease`.

$$[M, U, L] = plotdist(A, C, D1, D2, R, Z, upper, lower, ndraws, plotplease, legendplease)$$

The output M, U, L are $n \times (T + 1)$ matrices containing respectively the median and the upper and lower percentile of the smoothed distribution of X^T. `Upper` and `lower` should be of the form 0.975, 0.025 etc, `plotplease` and `legendplease` should be set to 1 if plots and legends are desired.

To use the code for the standard filter, simply set $D2$ equal to (the scalar) 0.