Acceleration of the order of convergence of a family of fractional fixed-point methods and its implementation in the solution of a nonlinear algebraic system related to hybrid solar receivers

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Abstract

This paper presents one way to define an uncountable family of fractional fixed-point methods through a set of matrices that can generate a group of fractional matrix operators, as well as one way to define groups of fractional operators that are isomorphic to the group of integers under the addition, and shows one way to classify and accelerate the order of convergence of the family of proposed iterative methods, which may be useful to continue expanding the applications of the fractional operators. The proposed method to accelerate the order of convergence is used in a fractional iterative method, and with the obtained method are solved simultaneously two nonlinear algebraic systems that depend on time-dependent parameters, and that allow obtaining the temperatures and efficiencies of a hybrid solar receiver. Finally, two uncountable families of fractional fixed-point methods are presented, in which the proposed method to accelerate convergence can be implemented.
Keywords: Fractional Operators; Group Theory; Order of Convergence; Fractional Iterative Methods

1. Introduction

In one dimension, a fractional derivative may be considered in a general way as a parametric operator of order α, such that it coincides with conventional derivatives when α is a positive integer n. So, when it is not necessary to explicitly specify the form of a fractional derivative, it is usually denoted as follows

$$\frac{d^\alpha}{dx^\alpha}.$$

On the other hand, a fractional differential equation is an equation that involves at least one differential operator of order α, with $(n-1) < \alpha \leq n$ for some positive integer n, and it is said to be a differential equation of order α if this operator is the highest order in the equation. The fractional operators have many representations, but one of their fundamental properties is that they allow retrieving the results of conventional calculus when $\alpha \to n$. So, considering a scalar function $h : \mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{1}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

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$$
\mathrm{O}_{x,\alpha}^{n}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha \to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{2}
$$

which may be proved to be a nonempty set through the following set of fractional operators

$$
\mathrm{O}_{0,x,\alpha}^{n}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) = \left(\partial_k^n + \mu(\alpha) \partial_k^\alpha \right) h(x) \ \text{ and } \ \lim_{\alpha \to n} \mu(\alpha) \partial_k^\alpha h(x) = 0 \ \forall k \geq 1 \right\}, \tag{3}
$$

whose complement may be defined as follows

$$
\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k \geq 1 \ \text{ and } \ \lim_{\alpha \to n} o_k^\alpha h(x) \neq \partial_k^n h(x) \ \text{ in at least one value } k \geq 1 \right\}, \tag{4}
$$

and which may be considered as a generating set of sets of **fractional tensor operators**. For example, considering $\alpha, n \in \mathbb{R}^d$ with $\alpha = \hat{e}_k [\alpha]_k$ and $n = \hat{e}_k [n]_k$, it is possible to define the following set of fractional tensor operators

$$
\mathrm{O}_{x,\alpha}^{n}(h) := \left\{ o_x^\alpha \ : \ \exists o_x^\alpha h(x) \ \text{ and } \ o_x^\alpha \in \mathrm{O}_{x,[\alpha]_1}^{[n]_1}(h) \times \mathrm{O}_{x,[\alpha]_2}^{[n]_2}(h) \times \cdots \times \mathrm{O}_{x,[\alpha]_d}^{[n]_d}(h) \right\}. \tag{5}
$$

One of the most famous fixed-point methods is the well-known Newton-Raphson method. However, it sometimes goes unnoticed that this method has the following problem related to finding roots of polynomials in the complex space: If it is necessary to find a complex root $\xi \in \mathbb{C} \setminus \mathbb{R}$ of a polynomial using the Newton-Raphson method, a complex initial condition x_0 must be provided, and if a suitable initial condition is selected, this will lead to a complex solution, but there is also the possibility that this may lead to a real solution. If the root obtained is real, it is necessary to change the initial condition and expect that this will lead to a complex solution, otherwise, it is necessary to change the value of the initial condition again, this process is repeated until it finally converges to a complex solution. The process described above is very similar to what happens when different values α are used in fractional operators until finding a solution that fulfills some established criterion.

Considering the Newton-Raphson method from the perspective of fractional calculus, it is possible to consider that an order α remains fixed, in this case $\alpha = 1$, and the initial conditions x_0 are varied until found a solution ξ that fulfills some established criterion. It is necessary to mention that considering a relationship between fractional calculus and the Newton-Raphson method may seem somewhat forced at first, but the latter is characterized by the fact that when it generates divergent sequences of complex numbers, it can sometimes lead to the creation of fractals [1], and this feature is complemented quite well with the fact that the orders of the fractional derivatives seem to be closely related to the fractal dimension [2]. Based on the above, it is possible to consider inverting the behavior of the order $\alpha = 1$ of the derivative and the initial condition x_0, that is, leaving the initial condition x_0 fixed and varying the order α of the derivative, thus obtaining the **fractional Newton-Raphson method** [3–5], which is nothing other than the Newton-Raphson method using any definition of a fractional operator that fits the function whose zeros want to be determined.

Before continuing, it is necessary to mention that due to the large number of fractional operators that may exist [6–17], some sets must be defined to fully characterize the fractional Newton-Raphson method. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as **fractional calculus of sets** [18]. So, considering a function $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following sets

$$
{}_m\mathrm{O}_{x,\alpha}^{n}(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{x,\alpha}^{n}\left([h]_k\right) \ \forall k \leq m \right\}, \tag{6}
$$

$$
{}_m\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{x,\alpha}^{n,c}\left([h]_k\right) \ \forall k \leq m \right\}, \tag{7}
$$

$$
{}_m\mathrm{O}_{x,\alpha}^{n,u}(h) := {}_m\mathrm{O}_{x,\alpha}^{n}(h) \cup {}_m\mathrm{O}_{x,\alpha}^{n,c}(h), \tag{8}
$$

where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$$
{}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h) := \bigcap_{k \in \mathbb{Z}} {}_m\mathrm{O}_{x,\alpha}^{k,u}(h), \tag{9}
$$

which under the classical Hadamard product it is fulfilled that

$$o_x^0 \circ h(x) := h(x)\ \forall o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h). \tag{10}$$

As a consequence, it is possible to define the following sets of matrices

$${}_m\mathrm{M}_{x,\alpha}^{\infty}(h) := \left\{A_{h,\alpha} = A_{h,\alpha}(o_x^\alpha)\ :\ o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)\ \text{ and }\ A_{h,\alpha}(x) = \left([A_{h,\alpha}]_{jk}(x)\right) := \left(o_k^\alpha[h]_j(x)\right)\right\}, \tag{11}$$

$${}_m\mathrm{IM}_{x,\alpha}^{\infty}(h) := \left\{A_{h,\alpha} \in {}_m\mathrm{M}_{x,\alpha}^{\infty}(h)\ :\ \exists A_{h,\alpha}^{-1}\right\}, \tag{12}$$

and therefore, the fractional Newton-Raphson method may be defined and classified through the set of matrices ${}_m\mathrm{IM}_{x,\alpha}^{\infty}(h)$ using the following set:

$$\left\{A_{h,\alpha}\ :\ \exists A_{h,\alpha}^{-1} \in {}_m\mathrm{IM}_{x,\alpha}^{\infty}(h)\ \text{ and }\ A_{h,\alpha}(x) = \left([A_{h,\alpha}]_{jk}(x)\right) := \left(o_k^\alpha[h]_j(x)\right)^{-1}\right\}. \tag{13}$$

Furthermore, considering that when using the classical Hadamard product in general $o_x^{p\alpha} \circ o_x^{q\alpha} \neq o_x^{(p+q)\alpha}$. Assuming the existence of a fixed set of matrices ${}_m\mathrm{IM}_{x,\alpha}^{\infty}(h)$, joined with a modified Hadamard product that fulfills the following property

$$o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha} := \begin{cases} o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o_{i,x}^{(p+q)\alpha}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{14}$$

by omitting the function h, the resulting set ${}_m\mathrm{IM}_{x,\alpha}^{\infty}(\cdot)$ has the ability to generate a group of **fractional matrix operators** A_α that fulfill the following equation

$$A_\alpha\left(o_{i,x}^{p\alpha}\right) \circ A_\alpha\left(o_{j,x}^{q\alpha}\right) := \begin{cases} A_\alpha\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), & \text{if } i \neq j \\ A_\alpha\left(o_{i,x}^{(p+q)\alpha}\right), & \text{if } i = j \end{cases}, \tag{15}$$

through the following set [18]:

$${}_m\mathrm{G}_{FNR}(\alpha) := \left\{A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right)\ :\ \exists A_\alpha^{\circ r} \in {}_m\mathrm{IM}_{x,\alpha}^{\infty}(\cdot)\ \forall r \in \mathbb{Z}\ \text{ and }\ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}. \tag{16}$$

Where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q} \in {}_m\mathrm{G}_{FNR}(\alpha)$, with $i \neq j$, the following property is defined

$$A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q} = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), \quad p, q \in \mathbb{Z} \setminus \{0\}, \tag{17}$$

as a consequence, it is fulfilled that

$$\forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FNR}(\alpha)\ \text{ such that }\ A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right)\ \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ(r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{18}$$

It is necessary to mention that for each operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ it is possible to define a group [19], which is isomorphic to the group of integers under the addition, as shown by the following theorems:

Theorem 1.1. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. So, considering the modified Hadamard product given by* (14)*, it is possible to define the following set of fractional matrix operators*

$${}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) := \left\{A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right)\ :\ r \in \mathbb{Z}\ \textit{ and }\ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \tag{19}$$

which corresponds to the Abelian group generated by the operator $A_\alpha\left(o_x^\alpha\right)$.

Proof. It should be noted that due to the way the set (19) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ it is fulfilled that

$$A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \left([A_\alpha^{\circ p}]_{jk}\right) \circ \left([A_\alpha^{\circ q}]_{jk}\right) = \left(o_k^{(p+q)\alpha}\right) = \left([A_\alpha^{\circ(p+q)}]_{jk}\right) = A_\alpha^{\circ(p+q)}, \tag{20}$$

with which it is possible to prove that the set ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ fulfills the following properties, which correspond to the properties of an Abelian group:

$$\begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q}, A_\alpha^{\circ r} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) \circ A_\alpha^{\circ r} = A_\alpha^{\circ p} \circ \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) \\ \exists A_\alpha^{\circ 0} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ such that } \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p} \\ \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ such that } A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0} \\ \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p} \end{cases}. \tag{21}$$

□

Theorem 1.2. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. So, the group generated by the operator $A_\alpha(o_x^\alpha)$ is isomorphic to the group $(\mathbb{Z},+)$, that is,*

$${}_mG(A_\alpha(o_x^\alpha)) \cong (\mathbb{Z},+). \tag{22}$$

Proof. To prove the theorem it is enough to define a bijective homomorphism between the sets ${}_mG(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$. Let $\psi : {}_mG(A_\alpha(o_x^\alpha)) \to (\mathbb{Z},+)$ be a function with inverse function $\psi^{-1} : (\mathbb{Z},+) \to {}_mG(A_\alpha(o_x^\alpha))$. So, the functions ψ and ψ^{-1} may be defined as follows

$$\psi(A_\alpha^{\circ r}) = r \quad \text{and} \quad \psi^{-1}(r) = A_\alpha^{\circ r}, \tag{23}$$

with which it is possible to obtain the following results:

$$\begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } \psi\left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) = \psi\left(A_\alpha^{\circ(p+q)}\right) = p+q = \psi\left(A_\alpha^{\circ p}\right) + \psi\left(A_\alpha^{\circ q}\right) \\ \forall p,q \in (\mathbb{Z},+) \text{ it is fulfilled that } \psi^{-1}(p+q) = A_\alpha^{\circ(p+q)} = A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \psi^{-1}(p) \circ \psi^{-1}(q) \end{cases}. \tag{24}$$

Therefore, from the previous results, it follows that the function ψ defines an isomorphism between the sets ${}_mG(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$.

□

Then, from the previous theorems it is possible to obtain the following corollaries:

Corollary 1.3. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. So, considering the modified Hadamard product given by (14) and some subgroup $\mathbb{H}$ of the group $(\mathbb{Z},+)$, it is possible to define the following set of fractional matrix operators*

$${}_mG(A_\alpha(o_x^\alpha), \mathbb{H}) := \left\{A_\alpha^{\circ r} = A_\alpha(o_x^{r\alpha}) \ : \ r \in \mathbb{H} \ \text{ and } \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \tag{25}$$

which corresponds to a subgroup of the group generated by the operator $A_\alpha(o_x^\alpha)$, that is,

$${}_mG(A_\alpha(o_x^\alpha), \mathbb{H}) \leq {}_mG(A_\alpha(o_x^\alpha)). \tag{26}$$

Corollary 1.4. *Let $h : \mathbb{R}^m \to \mathbb{R}^m$ be a function such that $\exists_m \mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. So, if it is fulfilled the following condition*

$$\forall o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h) \ \exists_m G(A_\alpha(o_x^\alpha)) \subset {}_m\mathrm{G}_{FNR}(\alpha), \tag{27}$$

such that ${}_mG(A_\alpha(o_x^\alpha))$ is the group generated by the operator $A_\alpha(o_x^\alpha)$. As a consequence, it is fulfilled that

$${}_m\mathrm{G}_{FNR}(\alpha) = \bigcup_{o_x^\alpha \in_m \mathrm{MO}_{x,\alpha}^{\infty,u}(h)} {}_mG(A_\alpha(o_x^\alpha)). \tag{28}$$

On the other hand, defining $A_\alpha(h) = \left([A_\alpha(h)]_{jk}\right) := \left([h]_k\right)$, it is possible to obtain the following result:

$$\forall A_\alpha^{\circ r} \in {}_m\mathrm{G}_{FNR}(\alpha)\ \exists A_{h,r\alpha} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h) \ \text{ such that } \ A_{h,r\alpha} := A_\alpha\left(o_x^{r\alpha}\right) \circ A_\alpha^T(h), \tag{29}$$

as a consequence, the fractional Newton-Raphson method may also be defined through the set of fractional matrix operators ${}_m\mathrm{G}_{FNR}(\alpha)$ using the following set:

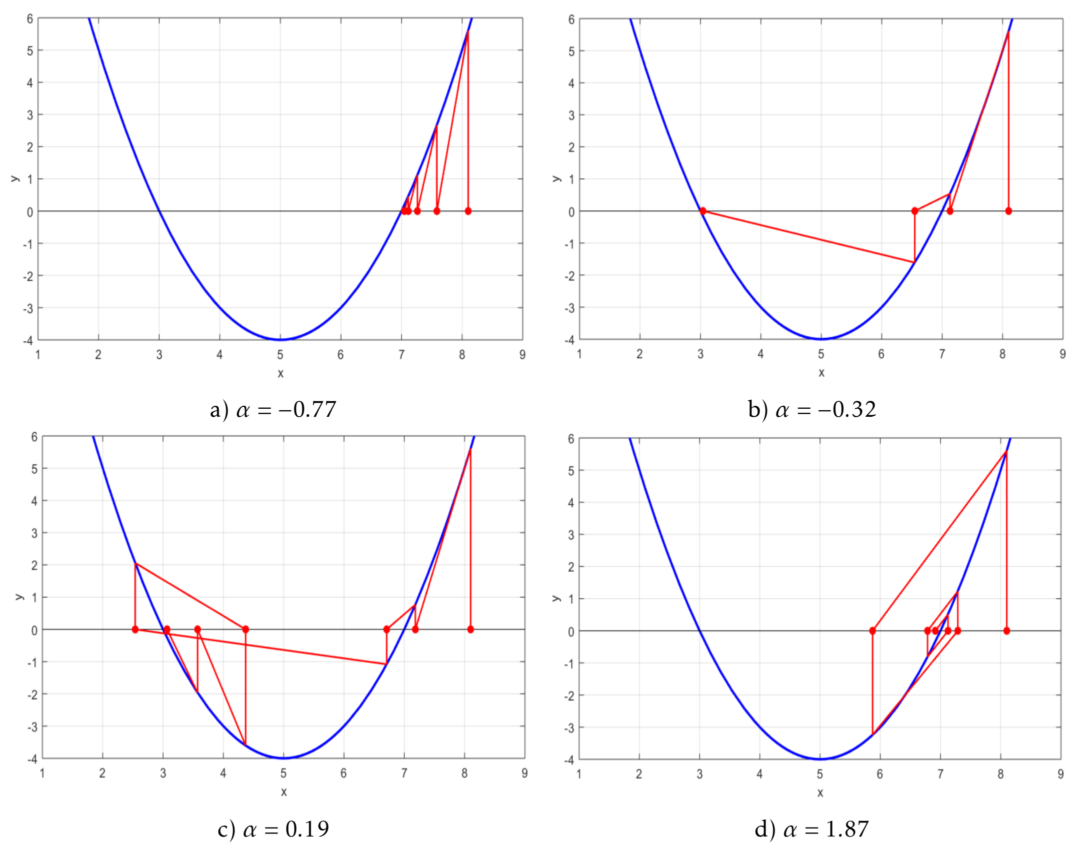
$$\left\{A_\alpha^{\circ 1} \in {}_m\mathrm{G}_{FNR}(\alpha)\ :\ \exists A_{h,\alpha}^{-1} = A_\alpha\left(o_x^{\alpha}\right) \circ A_\alpha^T(h) \ \text{ and } \ A_{h,\alpha}^{-1} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)\right\}. \tag{30}$$

Therefore, if Φ_{FNR} denotes the iteration function of the fractional Newton-Raphson method, it is possible to obtain the following results:

$$\text{Let } \alpha_0 \in \mathbb{R}\setminus\mathbb{Z} \ \Rightarrow\ \forall A_{h,\alpha_0}^{-1} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)\ \exists \Phi_{FNR} = \Phi_{FNR}(A_{h,\alpha_0})\ \therefore\ \forall A_{h,\alpha_0}\ \exists \left\{\Phi_{FNR}(A_{h,\alpha})\ :\ \alpha \in \mathbb{R}\setminus\mathbb{Z}\right\}, \tag{31}$$

$$\text{Let } \alpha_0 \in \mathbb{R}\setminus\mathbb{Z} \ \Rightarrow\ \forall A_{\alpha_0}^{\circ 1} \in {}_m\mathrm{G}_{FNR}(\alpha)\ \exists \Phi_{FNR} = \Phi_{FNR}(A_{\alpha_0})\ \therefore\ \forall A_{\alpha_0}\ \exists \left\{\Phi_{FNR}(A_{\alpha})\ :\ \alpha \in \mathbb{R}\setminus\mathbb{Z}\right\}. \tag{32}$$

The change from leaving the initial condition x_0 fixed and varying the order α of the fractional operators, although seemingly simple, gives the fractional Newton-Raphson method the ability to partially solve the intrinsic problem associated with classical fixed-point methods, which is that in general, to find N zeros of a function, N initial conditions must be provided. This is because by varying the order α of the fractional operators, the fractional Newton-Raphson method can find N zeros of a function using a single initial condition (see Figure 1). It is necessary to consider that mentioned above is also valid for any fixed-point method that implements fractional operators in some way, which may be named as fractional fixed-point methods or fractional iterative methods.



a) $\alpha = -0.77$ b) $\alpha = -0.32$ c) $\alpha = 0.19$ d) $\alpha = 1.87$

Figure 1: Illustrations of some trajectories generated by the fractional Newton-Raphson method for the same initial condition x_0 but with different orders α of the fractional operator used [3].

To finish this section, it is necessary to mention that the applications of fractional operators have spread to different fields of science such as finance [20, 21], economics [22], number theory through the Riemann zeta function [23, 24], and in engineering with the study for the manufacture of hybrid solar receivers [25, 26]. It is worth mentioning that there exists also a growing interest in fractional operators and their properties for solving nonlinear algebraic systems [18, 27–33], which is a classical problem in mathematics, physics and engineering, which consists of finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \ : \ \|f(\xi)\| = 0\}, \tag{33}$$

where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm, or equivalently

$$\{\xi \in \Omega \ : \ [f]_k(\xi) = 0 \ \forall k \geq 1\}. \tag{34}$$

Although finding the zeros of a function may seem like a simple problem, it is generally necessary to use numerical methods of the iterative type to solve it. So, considering that fractional iterative methods can find N solutions of a system using a single initial condition, this paper shows an alternative way to the Aitken's method to accelerate the order of convergence of a family of fractional fixed-point methods, which consists of implementing a function in the order of the fractional operators involved, with which it is possible to obtain an order of convergence (at least) quadratic.

2. Fixed-Point Method

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a function. It is possible to build a sequence $\{x_i\}_{i\geq 1}$ by defining the following iterative method

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2, \cdots. \tag{35}$$

So, if it is fulfilled that $x_i \to \xi \in \mathbb{R}^n$ and the function Φ is continuous around ξ, we obtain that

$$\xi = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \Phi(x_i) = \Phi\left(\lim_{i\to\infty} x_i\right) = \Phi(\xi), \tag{36}$$

the above result is the reason by which the method (35) is known as the **fixed-point method**. Furthermore, the function Φ is called an **iteration function**. On the other hand, considering the following set

$$B(\xi;\delta) := \{x \ : \ \|x - \xi\| < \delta\}, \tag{37}$$

it is possible to define the following corollary, which allows characterizing the order of convergence of an iteration function Φ through its **Jacobian matrix** $\Phi^{(1)}$ [5]:

Corollary 2.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, Φ has an* ***order of convergence*** *of order (at least) p in $B(\xi;\delta)$, where it is fulfilled that:*

$$p := \begin{cases} 1, & \text{if } \lim\limits_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| \neq 0 \\ 2, & \text{if } \lim\limits_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| = 0 \end{cases}. \tag{38}$$

3. Riemann-Liouville Fractional Operators

Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L^1_{loc}(a,b)$, where $L^1_{loc}(a,b)$ denotes the space of locally integrable functions on the open interval $(a,b) \subset \Omega$. One of the fundamental operators of fractional calculus is the operator **Riemann-Liouville fractional integral**, which is defined as follows [34, 35]

$$ {}_aI_x^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{39} $$

where Γ denotes the Gamma function. It is worth mentioning that the above operator is a fundamental piece to construct the operator **Riemann-Liouville fractional derivative**, which is defined as follows [34, 36]

$$ {}_aD_x^{\alpha} f(x) := \left\{ \begin{array}{cc} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ \dfrac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x) \right), & \text{if } \alpha \geq 0 \end{array} \right. , \tag{40} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f(x) := f(x)$. On the other hand, let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function n-times differentiable such that $f, f^{(n)} \in L^1_{loc}(a, b)$. Then, the Riemann-Liouville fractional integral also allows constructing the operator **Caputo fractional derivative**, which is defined as follows [34, 36]

$$ {}_a^CD_x^{\alpha} f(x) := \left\{ \begin{array}{cc} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ {}_aI_x^{n-\alpha} f^{(n)}(x), & \text{if } \alpha \geq 0 \end{array} \right. , \tag{41} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f^{(n)}(x) := f^{(n)}(x)$. Furthermore, if the function f fulfills that $f^{(k)}(a) = 0 \ \forall k \in \{0, 1, \cdots, n-1\}$, the Riemann-Liouville fractional derivative coincides with the Caputo fractional derivative, that is,

$$ {}_aD_x^{\alpha} f(x) = {}_a^CD_x^{\alpha} f(x). \tag{42} $$

So, applying the operator (40) with $a = 0$ to the function x^{μ}, with $\mu > -1$, we obtain the following result [5]:

$$ {}_0D_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{43} $$

where if $1 \leq \lceil \alpha \rceil \leq \mu$ it is fulfilled that ${}_0D_x^{\alpha} x^{\mu} = {}_0^CD_x^{\alpha} x^{\mu}$.

4. Fractional Fixed-Point Method

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$. So, considering an iteration function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, the iteration function of a fractional iterative method may be written in general form as follows

$$ \Phi(\alpha, x) := x - A_{g,\alpha}(x) f(x), \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{44} $$

where $A_{g,\alpha}$ is a matrix that depends, in at least one of its entries, on fractional operators of order α applied to some function $g : \mathbb{R}^n \to \mathbb{R}^n$, whose particular case occurs when $g = f$. So, it is possible to define in a general way a **fractional fixed-point method** as follows

$$ x_{i+1} := \Phi(\alpha, x_i), \quad i = 0, 1, 2, \cdots. \tag{45} $$

Before continuing, it is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i \geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$ \text{Conv}_{\delta}(\xi) := \left\{ \Phi \ : \ \lim_{x \to \xi} \Phi(\alpha, x) = \xi_{\alpha} \in B(\xi; \delta) \right\}, \tag{46} $$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i\geq 1}$ to some value $\xi_\alpha \in B(\xi;\delta)$. So, denoting by card$(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [18], proof of **Theorem** 2):

$$\operatorname{card}\left(\operatorname{Conv}_\delta(\xi)\right) = \operatorname{card}(\mathbb{R}), \tag{47}$$

from which it follows that the set (46) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following proposition [5]:

Proposition 4.1. *Let $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{R}^n \to \mathbb{R}^n$ be an iteration function such that $\Phi \in \operatorname{Conv}_\delta(\xi)$ in a region Ω. So, if Φ is given by the equation* (44) *and fulfills the following condition*

$$\lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi)\right)^{-1}. \tag{48}$$

Then, Φ fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

Proof. If Φ is given by the equation (44), the k-th component of the function Φ may be written as follows

$$[\Phi]_k(\alpha,x) = [x]_k - \sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(x)[f]_j(x), \tag{49}$$

and considering that $f^{(1)}(x) = \left([f^{(1)}]_{jl}(x)\right) := \left(\partial_l[f]_j(x)\right)$, it is possible to obtain the following result

$$[\Phi^{(1)}]_{kl}(\alpha,x) = \partial_l[\Phi]_k(\alpha,x) = \delta_{kl} - \sum_{j=1}^{n}\left([A_{g,\alpha}]_{kj}(x)[f^{(1)}]_{jl}(x) + \left(\partial_l[A_{g,\alpha}]_{kj}(x)\right)[f]_j(x)\right),$$

where δ_{kl} denotes the Kronecker delta. On the other hand, since f has a point $\xi\in\Omega$ such that $\|f(\xi)\| = 0$, it follows that

$$[\Phi^{(1)}]_{kl}(\alpha,\xi) = \delta_{kl} - \sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(\xi)[f^{(1)}]_{jl}(\xi).$$

Then, if $\Phi\in\operatorname{Conv}_\delta(\xi)$ and has an order of convergence (at least) quadratic in $B(\xi;\delta)$, by the **Corollary 2.1**, it is fulfilled the following condition

$$\sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(\xi)[f^{(1)}]_{jl}(\xi) = \delta_{kl}, \quad \forall k,l\leq n, \tag{50}$$

which may be rewritten more compactly as follows

$$A_{g,\alpha}(\xi)f^{(1)}(\xi) = I_n,$$

where I_n denotes the identity matrix of $n\times n$. Therefore, any matrix $A_{g,\alpha}$ that fulfills the following condition

$$\lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi)\right)^{-1},$$

ensures that the iteration function Φ given by the equation (44), fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

□

Considering the **Corollary 2.1** and the **Proposition 4.1**, it is possible to define the following sets to classify the order of convergence of some fractional iterative methods:

$$\operatorname{Ord}^1(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} \left\| \Phi^{(1)}(\alpha, x) \right\| \neq 0 \right\}, \tag{51}$$

$$\operatorname{Ord}^2(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} \left\| \Phi^{(1)}(\alpha, x) \right\| = 0 \right\}, \tag{52}$$

$$\operatorname{ord}^1(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} A_{g,\alpha}(x) \neq \left(f^{(1)}(\xi) \right)^{-1} \ \text{ or } \ \lim_{\alpha\to 1} A_{g,\alpha}(\xi) \neq \left(f^{(1)}(\xi) \right)^{-1} \right\}, \tag{53}$$

$$\operatorname{ord}^2(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi) \right)^{-1} \ \text{ or } \ \lim_{\alpha\to 1} A_{g,\alpha}(\xi) = \left(f^{(1)}(\xi) \right)^{-1} \right\}. \tag{54}$$

On the other hand, considering that depending on the nature of the function f, there exist cases in which the Newton-Raphson method can present an order of convergence (at least) linear [5]. So, it is possible to obtain the following relations between the previous sets

$$\operatorname{ord}^1(\xi) \subset \operatorname{Ord}^1(\xi) \quad \text{and} \quad \operatorname{ord}^2(\xi) \subset \operatorname{Ord}^1(\xi) \cup \operatorname{Ord}^2(\xi), \tag{55}$$

with which it is possible to define the following sets

$$\operatorname{Ord}_2^1(\xi) := \operatorname{ord}^2(\xi) \cap \operatorname{Ord}^1(\xi) \quad \text{and} \quad \operatorname{Ord}_2^2(\xi) := \operatorname{ord}^2(\xi) \cap \operatorname{Ord}^2(\xi). \tag{56}$$

4.1. Acceleration of the Order of Convergence of the Set $\operatorname{Ord}_2^1(\xi)$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and denoting by Φ_{NR} to the iteration function of the Newton-Raphson method, it is possible to define the following set of functions

$$\operatorname{Ord}_{NR}^2(\xi) := \left\{ f \ : \ \lim_{x\to\xi} \left\| \Phi_{NR}^{(1)}(x) \right\| = 0 \right\}. \tag{57}$$

So, it is possible to define the following corollary:

Corollary 4.2. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $f \in \operatorname{Ord}_{NR}^2(\xi)$, and let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function given by the equation (44) such that $\Phi \in \operatorname{ord}^1(\xi)$. So, if Φ also fulfills the following condition*

$$\lim_{\alpha\to 1} A_{g,\alpha}(\xi) = \left(f^{(1)}(\xi) \right)^{-1}. \tag{58}$$

Then, $\Phi \in \operatorname{Ord}_2^1(\xi)$. Therefore, it is possible to assign a positive value δ_0, and replace the order α of the fractional operators of the matrix $A_{g,\alpha}$ by the following function

$$\alpha_f([x]_k, x) := \begin{cases} \alpha, & \text{if } \ \|[x]_k| \neq 0 \ \text{ and } \ \|f(x)\| > \delta_0 \\ 1, & \text{if } \ \|[x]_k| = 0 \ \text{ or } \ \|f(x)\| \leq \delta_0 \end{cases}, \tag{59}$$

obtaining a new matrix that may be denoted as follows

$$A_{g,\alpha_f}(x) = \left([A_{g,\alpha_f}]_{jk}(x) \right), \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{60}$$

and with which it is fulfilled that $\Phi \in \operatorname{Ord}_2^2(\xi)$.

It is necessary to mention that the origin of the function (59) arises from the need to accelerate the order of convergence of the fractional Newton-Raphson method, which generated the method known as the **fractional Newton method**, whose matrix A_{g,α_f} corresponds to a particular case in which $g = f$ [3–5]. Finally, for practical purposes, it may be defined that if a fractional iterative method Φ fulfills the properties of the **Corollary 4.2** and uses the function (59), it may be called a **fractional iterative method accelerated**. It is worth mentioning that if $\Phi \in \text{Conv}_\delta(\xi)$, it is possible to obtain a numerical estimate of its order of convergence through the following corollary [19]:

Corollary 4.3. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function such that $\Phi \in \text{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi; 1/2)$, for some $m \in \mathbb{N}$, there exists a sequence $\{P_i\}_{i\geq m} \in B(p; \delta_K)$ given by the following values*

$$P_i := \frac{\log(\|x_i - x_{i-1}\|)}{\log(\|x_{i-1} - x_{i-2}\|)}, \tag{61}$$

such that it fulfills the following condition:

$$\lim_{i\to\infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_k \in B(p; \epsilon). \tag{62}$$

On the other hand, it should be noted that if $\Phi \in \text{Conv}_\delta(\xi)$ and $\|f(\xi)\| = 0$, it is fulfilled that

$$\lim_{i\to\infty} \|x_i - x_{i+1}\| \approx \lim_{i\to\infty} \|f(x_i)\|,$$

and as a consequence, it is possible to define the following corollary, which is useful for cases in which it is not possible to apply the **Corollary 4.3**:

Corollary 4.4. *Let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function given by the equation (44) such that $\Phi \in \text{Conv}_\delta(\xi)$. So, if Φ has an order of convergence of order (at least) p in $B(\xi; 1/2)$, for some $m \in \mathbb{N}$, there exists a sequence $\left\{P_{f,i}\right\}_{i\geq m} \in B(p; \delta_K)$ given by the following values*

$$P_{f,i} := \frac{\log(\|f(x_i)\|)}{\log(\|f(x_{i-1})\|)}, \tag{63}$$

such that it fulfills the following condition:

$$\lim_{i\to\infty} P_{f,i} \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_{f,k} \in B(p; \epsilon). \tag{64}$$

5. Equations of a Hybrid Solar Receiver

Considering the notation

$$s = (T_{cell}, T_{hot}, T_{cold}, \eta_{cell}, \eta_{TEG})^T := ([x]_1, [x]_2, [x]_3, [x]_4, [x]_5)^T,$$

the following expressions

$$\begin{cases} a_0 = \dfrac{2r_{intercon}}{\sqrt{f^* A_{TEG}}\left(b\sqrt{f^*}+\sqrt{A_{TEG}}\right)}, \quad a_1 = \eta_{opt} C_g DNI, \quad a_2 = r_{cell} + r_{sol} + A_{cell}\left(\dfrac{r_{cop}+r_{cer}}{A_{TEG}} + a_0\right) \\ a_3 = \dfrac{A_{cell} l}{f^* A_{TEG} k_{TEG}}, \quad a_4 = T_{air}, \quad a_5 = A_{cell}\left(\dfrac{r_{cer}}{A_{TEG}} + R_{heat\_exch} + a_0\right), \quad a_6 = -\eta_{cell,ref}\gamma_{cell} \\ a_7 = \eta_{cell,ref}\left(1+25\gamma_{cell}\right), \quad a_8 = \sqrt{1+ZT}, \quad a_9 = 273.15 \end{cases},$$

and the following particular values [37]:

$$\begin{cases} \eta_{opt} = 0.85, & r_{intercon} = 2.331\times 10^{-7}, & C_g = 800 \\ A_{cell} = 9\times 10^{-6}, & R_{heat\_exch} = 0.5, & A_{TEG} = 5.04\times 10^{-5} \\ \eta_{cell,ref} = 0.43, & r_{cell} = 3\times 10^{-6}, & f^* = 0.7 \\ \gamma_{cell} = 4.6\times 10^{-4}, & r_{sol} = 1.603\times 10^{-6}, & b = 5\times 10^{-4} \\ r_{cop} = 7.5\times 10^{-7}, & r_{cer} = 8\times 10^{-6}, & l = 5\times 10^{-4} \\ k_{TEG} = 1.5, & ZT = 1 & \end{cases}.$$

It is possible to define the following system of equations that corresponds to the combination of a solar photovoltaic system with a thermoelectric generator system [38, 39], which is named as a **hybrid solar receiver**

$$\begin{cases} [x]_1 = [x]_2 + a_1 a_2 \left(1-[x]_4\right) \\ [x]_2 = [x]_3 + a_1 a_3 \left(1-[x]_4\right)\left(1-[x]_5\right) \\ [x]_3 = a_4 + a_1 a_5 \left(1-[x]_4\right)\left(1-[x]_5\right) \\ [x]_4 = a_6 [x]_1 + a_7 \\ [x]_5 = \left(a_8 - 1\right)\left(1 - \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)\left(a_8 + \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)^{-1} \end{cases}, \tag{65}$$

whose deduction, as well as details about its interpretation, may be found in the reference [37]. Using the system of equations (65), it is possible to define a function $f_1 : \Omega \subset \mathbb{R}^5 \to \mathbb{R}^5$, that is,

$$f_1(s) := \begin{pmatrix} [x]_1 - [x]_2 - a_1 a_2 \left(1-[x]_4\right) \\ [x]_2 - [x]_3 - a_1 a_3 \left(1-[x]_4\right)\left(1-[x]_5\right) \\ [x]_3 - a_4 - a_1 a_5 \left(1-[x]_4\right)\left(1-[x]_5\right) \\ [x]_4 - a_6 [x]_1 - a_7 \\ [x]_5 - \left(a_8 - 1\right)\left(1 - \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)\left(a_8 + \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)^{-1} \end{pmatrix}, \tag{66}$$

which depends on two parameters, the direct normal irradiance (DNI) and the ambient temperature (T_{air}). These parameters are measured in real-time at certain times of the day [37], and it is necessary to calculate a new solution of the system (65) for each new pair of parameters, that is,

$$(DNI, T_{air}) \xrightarrow{f_1} s \in \mathbb{R}^5.$$

However, to simplify the task of finding the solutions of the system (65), it is possible through the consecutive substitution of the variables $[x]_1$, $[x]_4$, $[x]_5$ and some algebraic simplifications, to obtain the following transcendental system [26]:

$$\begin{cases} [x]_2 = [x]_3 - a_1 a_3 \dfrac{\left(a_6 [x]_2 + a_7 - 1\right)\left(a_8\left([x]_3 + a_9\right) + \left([x]_2 + a_9\right)\right)}{\left(1 + a_1 a_2 a_6\right)\left(a_8\left([x]_2 + a_9\right) + \left([x]_3 + a_9\right)\right)} \\ [x]_3 = a_4 - a_1 a_5 \dfrac{\left(a_6 [x]_2 + a_7 - 1\right)\left(a_8\left([x]_3 + a_9\right) + \left([x]_2 + a_9\right)\right)}{\left(1 + a_1 a_2 a_6\right)\left(a_8\left([x]_2 + a_9\right) + \left([x]_3 + a_9\right)\right)} \end{cases}, \tag{67}$$

whose solution allows knowing the values of the variables $[x]_1, [x]_4$ and $[x]_5$ through the following equations

$$
\begin{cases}
[x]_1 = \dfrac{[x]_2 - a_1 a_2 (a_7 - 1)}{1 + a_1 a_2 a_6} \\
[x]_4 = \dfrac{a_6 (a_1 a_2 + [x]_2) + a_7}{1 + a_1 a_2 a_6} \\
[x]_5 = \dfrac{(a_8 - 1)([x]_2 - [x]_3)}{a_8 ([x]_2 + a_9) + ([x]_3 + a_9)}
\end{cases}. \tag{68}
$$

Using the system of equations (67), it is possible to define a function $f_2 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, that is,

$$
f_2(x) := \begin{pmatrix} [x]_2 - [x]_3 + a_1 a_3 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \\ [x]_3 - a_4 + a_1 a_5 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \end{pmatrix}, \tag{69}
$$

and then finding the solutions of the function (69), through the equations (68), it is possible to construct the solutions of the function (66).

5.1. Solutions of the Equations of a Hybrid Solar Receiver

To solve the equation (69) and at the same time solve the equation (66), a fractional fixed-point method will be used, as well as its accelerated version through the function (59). Before continuing, it is necessary to mention that for some definitions of fractional operators it is fulfilled that the derivative of order α of a constant is different from zero (for example: Riesz, Grünwald–Letnikov, Riemann-Liouville, etc. [34–36]), that is,

$$
\partial_k^\alpha c := \frac{\partial^\alpha}{\partial [x]_k^\alpha} c \neq 0, \quad c = constant. \tag{70}
$$

So, considering a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, the Riemann-Liouville fractional derivative given by the equation (43), and an iteration function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, it is possible to define the following fractional fixed-point method

$$
x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{g_f,\beta}(x_i) f(x_i), \quad i = 0, 1, 2, \cdots, \tag{71}
$$

where $A_{g_f,\beta}(x_i)$ is given by the following expression

$$
A_{g_f,\beta}(x_i) = \left([A_{g_f,\beta}]_{jk}(x_i)\right) := \left(\partial_k^{\beta(\alpha,[x_i]_k)} [g_f]_j(x)\right)_{x_i}^{-1}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{72}
$$

with $g_f(x)$ and $\beta(\alpha, [x_i]_k)$ functions defined as follows

$$
g_f(x) := f(x_i) + f^{(1)}(x_i) x \quad \text{and} \quad \beta(\alpha, [x_i]_k) := \begin{cases} \alpha, & \text{if} \quad |[x_i]_k| \neq 0 \\ 1, & \text{if} \quad |[x_i]_k| = 0 \end{cases}. \tag{73}
$$

The fractional iterative method given by the equation (71) is named the **fractional quasi-Newton method**. On the other hand, if it is assumed that $\Phi \in \mathrm{Conv}_\delta(\xi)$, then it is fulfilled that $\Phi \in \mathrm{ord}^1(\xi)$. Furthermore, the method fulfills the following condition

$$
\lim_{\alpha \to 1} \partial_k^{\beta(\alpha,[x_i]_k)} [g_f]_j(x_i) = \partial_k [f]_j(x_i), \quad 1 \leq j, k \leq n, \tag{74}
$$

and as a consequence $\Phi \in \mathrm{Ord}_2^1(\xi)$. So, if it is assumed that $f \in \mathrm{Ord}_{NR}^2(\xi)$, by the **Corollary 4.2**, it is possible to construct the **fractional quasi-Newton method accelerated** using the following matrix

$$A_{g_f,\alpha_f}(x_i) = \left([A_{g_f,\alpha_f}]_{jk}(x_i)\right) := \left(\partial_k^{\alpha_f([x_i]_k,x_i)}[g_f]_j(x)\right)_{x_i}^{-1}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}. \tag{75}$$

Before continuing, it is necessary to mention that a description of the algorithm that must be implemented when working with a fractional iterative method given by the equation (45) may be found in the reference [4]. On the other hand, a simplified example of how the methods given by the matrices (72) and (75) should be programmed may be found in the reference [19]. Using the fractional fixed-point methods defined by the matrices (72) and (75), we proceed to find three solutions of the function (69) keeping fixed the following values:

$$\delta_0 = 13 \quad \text{and} \quad x_0 = (3000, 3000)^T.$$

Example 1. *Considering by hypothesis that $f_2 \in \mathrm{Ord}_{NR}^2(\xi)$, and using the following values*

$$DNI = 900, \quad T_{air} = 20, \quad \alpha = 0.89825,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (72) *and* (75).

*i) If Φ use $A_{g_f,\beta} \Rightarrow \Phi \in \mathrm{Ord}_2^1(\xi)$. On the other hand, from **Table 1** and **Corollary 4.3**, $P_{27} \approx 1.09 \in B(p;\delta_K)$, which is consistent with **Corollary 2.1**, since in general $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, which follows from the proof of **Proposition 4.1**. So, it is concluded that Φ has an order of convergence (at least) linear, that is, $\Phi \in \mathrm{Ord}^1(\xi)$.*

Table 1: Iterations generated by the fractional quasi-Newton method.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2048.526273	2036.688326	1.35E+03	2.01E+03	2052.245932	0.02901075	0.00087668	2.01E+03
2	1378.380727	1357.837031	9.54E+02	1.33E+03	1381.592211	0.16166606	0.00214528	1.33E+03
3	914.5756647	887.7554749	6.60E+02	8.65E+02	917.4354426	0.25347627	0.00391089	8.65E+02
4	599.7868499	568.5654338	4.48E+02	5.46E+02	602.4079218	0.31578871	0.00622874	5.46E+02
5	390.7721777	356.5990844	2.98E+02	3.34E+02	393.2347526	0.35716317	0.0090235	3.34E+02
6	255.3927888	219.4044444	1.93E+02	1.97E+02	257.7527048	0.38396151	0.0120214	1.97E+02
7	170.1536777	133.2761535	1.21E+02	1.11E+02	172.4489564	0.4008346	0.01478215	1.11E+02
8	118.188164	81.23045449	7.35E+01	5.95E+01	120.4440369	0.41112117	0.01686287	5.95E+01
9	87.62585188	51.33933793	4.27E+01	2.99E+01	89.85854925	0.41717098	0.01800683	2.99E+01
10	70.31181026	35.35034092	2.36E+01	1.43E+01	72.5313783	0.42059829	0.01823343	1.43E+01
11	60.85889363	27.58761689	1.22E+01	6.66E+00	63.07129347	0.4224695	0.01782623	6.66E+00
12	55.92035933	24.22121073	5.98E+00	3.07E+00	58.12901425	0.42344708	0.01721438	3.07E+00
13	53.49709311	22.89305436	2.76E+00	1.38E+00	55.70391046	0.42392677	0.01672394	1.38E+00
14	52.38726485	22.39252245	1.22E+00	6.02E-01	54.59324061	0.42414646	0.01643587	6.02E-01
15	51.90534374	22.20463447	5.17E-01	2.55E-01	54.11095406	0.42424185	0.01629349	2.55E-01
16	51.70286627	22.13313077	2.15E-01	1.06E-01	53.90832305	0.42428193	0.01622933	1.06E-01
17	51.61937072	22.10548244	8.80E-02	4.32E-02	53.82476418	0.42429846	0.01620181	4.32E-02
18	51.58529753	22.09465371	3.58E-02	1.76E-02	53.79066515	0.42430521	0.01619031	1.76E-02
19	51.5714752	22.09037372	1.45E-02	7.12E-03	53.77683234	0.42430794	0.01618559	7.12E-03
20	51.56588734	22.0886717	5.84E-03	2.87E-03	53.77124024	0.42430905	0.01618366	2.87E-03
21	51.56363304	22.08799215	2.35E-03	1.16E-03	53.76898424	0.42430949	0.01618288	1.16E-03
22	51.56272473	22.08772013	9.48E-04	4.67E-04	53.76807524	0.42430967	0.01618256	4.67E-04
23	51.56235904	22.08761106	3.82E-04	1.88E-04	53.76770927	0.42430975	0.01618243	1.88E-04
24	51.56221188	22.08756728	1.54E-04	7.56E-05	53.767562	0.42430978	0.01618238	7.58E-05
25	51.56215268	22.0875497	6.18E-05	3.04E-05	53.76750275	0.42430979	0.01618236	3.05E-05
26	51.56212886	22.08754263	2.48E-05	1.22E-05	53.76747891	0.42430979	0.01618235	1.20E-05
27	51.56211928	22.08753979	9.99E-06	4.92E-06	53.76746933	0.42430979	0.01618235	4.72E-06

*ii) If Φ use $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}_2^2(\xi)$. On the other hand, from **Table 2** and **Corollary 4.4**, $P_{f,13} \approx 2.81 \in B(p;\delta_K)$, which is consistent with **Corollary 2.1**, since in general $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, which follows from the proof of **Proposition 4.1**. So, it is concluded that Φ has an order of convergence (at least) quadratic, that is, $\Phi \in \mathrm{Ord}^2(\xi)$.*

Table 2: Iterations generated by the fractional quasi-Newton method accelerated.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2048.526273	2036.688326	1.35E+03	2.01E+03	2052.245932	0.02901075	0.00087668	2.01E+03
2	1378.380727	1357.837031	9.54E+02	1.34E+03	1381.592211	0.16166606	0.00214528	1.34E+03
3	914.5756647	887.7554749	6.60E+02	8.65E+02	917.4354426	0.25347627	0.00391089	8.65E+02
4	599.7868499	568.5654338	4.48E+02	5.46E+02	602.4079218	0.31578871	0.00622874	5.46E+02
5	390.7721777	356.5990844	2.98E+02	3.34E+02	393.2347526	0.35716317	0.0090235	3.34E+02
6	255.3927888	219.4044444	1.93E+02	1.97E+02	257.7527048	0.38396151	0.0120214	1.97E+02
7	170.1536777	133.2761535	1.21E+02	1.11E+02	172.4489564	0.4008346	0.01478215	1.11E+02
8	118.188164	81.23045449	7.36E+01	5.95E+01	120.4440369	0.41112117	0.01686287	5.95E+01
9	87.62585188	51.33933793	4.28E+01	2.99E+01	89.85854925	0.41717098	0.01800683	2.99E+01
10	70.31181026	35.35034092	2.36E+01	1.43E+01	72.5313783	0.42059829	0.01823343	1.43E+01
11	60.85889363	27.58761689	1.22E+01	6.66E+00	63.07129347	0.4224695	0.01782623	6.66E+00
12	51.56100988	22.08746493	1.08E+01	1.04E-03	53.76635909	0.42431001	0.01618182	1.04E-03
13	51.56211284	22.08753788	1.11E-03	4.13E-09	53.76746288	0.4243098	0.01618235	3.03E-07

Example 2. *Considering by hypothesis that $f_2 \in \mathrm{Ord}^2_{NR}(\xi)$, and using the following values*

$$DNI = 574.319, \quad T_{air} = 16.832, \quad \alpha = 0.8996,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (72) *and* (75).

*i) If Φ use $A_{g_f,\beta} \Rightarrow \Phi \in \mathrm{Ord}^1_2(\xi)$. On the other hand, from **Table 3** and **Corollary 4.3**, $P_{26} \approx 1.09 \in B(p;\delta_K)$, which is consistent with **Corollary 2.1**, since in general $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, which follows from the proof of **Proposition 4.1**. So, it is concluded that Φ has an order of convergence (at least) linear, that is, $\Phi \in \mathrm{Ord}^1(\xi)$.*

Table 3: Iterations generated by the fractional quasi-Newton method.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2029.854772	2022.247443	1.38E+03	2.00E+03	2032.218723	0.03297214	0.00056752	2.00E+03
2	1351.035349	1337.861649	9.64E+02	1.32E+03	1353.07091	0.16730757	0.00139631	1.32E+03
3	884.5286725	867.3584839	6.63E+02	8.49E+02	886.3385526	0.25962723	0.00256042	8.49E+02
4	570.3098992	550.3428213	4.46E+02	5.32E+02	571.9677708	0.32180977	0.00410184	5.32E+02
5	363.4003476	341.5519319	2.94E+02	3.23E+02	364.9581233	0.36275628	0.00597385	3.23E+02
6	230.608119	207.585416	1.89E+02	1.89E+02	232.1016543	0.38903529	0.00799251	1.89E+02
7	147.8561494	124.2274595	1.18E+02	1.06E+02	149.3096521	0.40541155	0.0098586	1.06E+02
8	98.01126796	74.27302588	7.06E+01	5.63E+01	99.44065736	0.41527564	0.01127184	5.63E+01
9	69.13937735	45.768905	4.06E+01	2.79E+01	70.5547995	0.42098926	0.01205541	2.79E+01
10	53.13057813	30.57994962	2.21E+01	1.29E+01	54.53825576	0.42415733	0.01220761	1.29E+01
11	44.6597286	23.22992376	1.12E+01	5.69E+00	46.06330831	0.42583368	0.01190151	5.69E+00
12	40.41192388	20.07409231	5.29E+00	2.44E+00	41.81344865	0.4266743	0.01143556	2.44E+00
13	38.42463196	18.85806286	2.33E+00	1.02E+00	39.82519534	0.42706758	0.01106236	1.02E+00
14	37.56159752	18.41580876	9.70E-01	4.16E-01	38.9617434	0.42723837	0.01084908	4.16E-01
15	37.20752766	18.25657936	3.88E-01	1.65E-01	38.60750225	0.42730844	0.01074838	1.65E-01
16	37.06714943	18.19864095	1.52E-01	6.44E-02	38.46705611	0.42733622	0.01070538	6.44E-02
17	37.01251861	18.17727243	5.87E-02	2.49E-02	38.41239886	0.42734703	0.01068795	2.49E-02
18	36.99146859	18.16930635	2.25E-02	9.54E-03	38.39133866	0.42735119	0.01068108	9.54E-03
19	36.98340172	18.16631458	8.60E-03	3.65E-03	38.38326789	0.42735279	0.01067841	3.65E-03
20	36.98031974	18.16518558	3.28E-03	1.39E-03	38.38018441	0.4273534	0.01067738	1.39E-03
21	36.97914434	18.16475823	1.25E-03	5.31E-04	38.37900845	0.42735363	0.01067699	5.30E-04
22	36.97869653	18.16459616	4.76E-04	2.02E-04	38.37856042	0.42735372	0.01067684	2.02E-04
23	36.97852602	18.16453462	1.81E-04	7.69E-05	38.37838983	0.42735375	0.01067678	7.67E-05
24	36.97846112	18.16451124	6.90E-05	2.93E-05	38.3783249	0.42735377	0.01067676	2.93E-05
25	36.97843643	18.16450235	2.62E-05	1.11E-05	38.37830019	0.42735377	0.01067675	1.10E-05
26	36.97842703	18.16449898	9.99E-06	4.23E-06	38.37829079	0.42735377	0.01067675	4.12E-06

*ii) If Φ use $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}^2_2(\xi)$. On the other hand, from **Table 4** and **Corollary 4.4**, $P_{f,12} \approx 2.77 \in B(p;\delta_K)$, which is consistent with **Corollary 2.1**, since in general $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, which follows from the proof of **Proposition 4.1**. So, it is concluded that Φ has an order of convergence (at least) quadratic, that is, $\Phi \in \mathrm{Ord}^2(\xi)$.*

Table 4: Iterations generated by the fractional quasi-Newton method accelerated.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2029.854772	2022.247443	1.38E+03	2.00E+03	2032.218723	0.03297214	0.00056752	2.00E+03
2	1351.035349	1337.861649	9.64E+02	1.32E+03	1353.07091	0.16730757	0.00139631	1.32E+03
3	884.5286725	867.3584839	6.63E+02	8.49E+02	886.3385526	0.25962723	0.00256042	8.49E+02
4	570.3098992	550.3428213	4.46E+02	5.32E+02	571.9677708	0.32180977	0.00410184	5.32E+02
5	363.4003476	341.5519319	2.94E+02	3.23E+02	364.9581233	0.36275628	0.00597385	3.23E+02
6	230.608119	207.585416	1.89E+02	1.89E+02	232.1016543	0.38903529	0.00799251	1.89E+02
7	147.8561494	124.2274595	1.18E+02	1.06E+02	149.3096521	0.40541155	0.0098586	1.06E+02
8	98.01126796	74.27302588	7.06E+01	5.63E+01	99.44065736	0.41527564	0.01127184	5.63E+01
9	69.13937735	45.768905	4.06E+01	2.79E+01	70.5547995	0.42098926	0.01205541	2.79E+01
10	53.13057813	30.57994962	2.21E+01	1.29E+01	54.53825576	0.42415733	0.01220761	1.29E+01
11	36.97715447	18.16441312	2.04E+01	1.19E-03	38.37701761	0.42735403	0.0106761	1.19E-03
12	36.97842127	18.1644969	1.27E-03	7.75E-09	38.37828503	0.42735378	0.01067675	2.15E-07

Example 3. *Considering by hypothesis that $f_2 \in \mathrm{Ord}^2_{NR}(\xi)$, and using the following values*

$$DNI = 94.3555, \quad T_{air} = 8.373, \quad \alpha = 0.89914,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (72) *and* (75).

i) If Φ use $A_{g_f,\beta} \Rightarrow \Phi \in \mathrm{Ord}^1_2(\xi)$. On the other hand, from ***Table 5*** *and* ***Corollary 4.3****, $P_{25} \approx 1.1 \in B(p;\delta_K)$, which is consistent with* ***Corollary 2.1****, since in general $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, which follows from the proof of* ***Proposition 4.1****. So, it is concluded that Φ has an order of convergence (at least) linear, that is, $\Phi \in \mathrm{Ord}^1(\xi)$.*

Table 5: Iterations generated by the fractional quasi-Newton method.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2026.628123	2025.393793	1.38E+03	2.02E+03	2027.016086	0.03400122	0.00009211	2.02E+03
2	1343.689808	1341.54807	9.66E+02	1.33E+03	1344.023514	0.16909715	0.0002274	1.33E+03
3	872.9738032	870.1756552	6.66E+02	8.62E+02	873.2701122	0.26221217	0.0004193	8.62E+02
4	554.9152667	551.6514102	4.50E+02	5.43E+02	555.1863071	0.32512915	0.00067737	5.43E+02
5	344.7596997	341.173768	2.97E+02	3.33E+02	345.014044	0.36670122	0.00099809	3.33E+02
6	209.3841806	205.5840939	1.92E+02	1.97E+02	209.6277698	0.39348063	0.0013556	1.97E+02
7	124.6869147	120.7550587	1.20E+02	1.12E+02	124.9237751	0.41023508	0.00170264	1.12E+02
8	73.46486534	69.46807813	7.25E+01	6.09E+01	73.69765627	0.4203676	0.00198789	6.09E+01
9	43.70741877	39.70792607	4.21E+01	3.11E+01	43.93784558	0.42625409	0.00217704	3.11E+01
10	27.24135725	23.30847331	2.32E+01	1.47E+01	27.47047589	0.42951134	0.00225857	1.47E+01
11	18.66834592	14.88366691	1.20E+01	6.33E+00	18.89678346	0.43120722	0.0022372	6.33E+00
12	14.54219733	10.9762278	5.68E+00	2.43E+00	14.77030707	0.43202343	0.00213764	2.43E+00
13	12.74609214	9.40115093	2.39E+00	8.48E-01	12.97405918	0.43237873	0.00201715	8.48E-01
14	12.04816377	8.85012383	8.89E-01	2.80E-01	12.27607536	0.43251679	0.00193289	2.80E-01
15	11.80217299	8.67286234	3.03E-01	9.00E-02	12.03006504	0.43256545	0.0018928	9.00E-02
16	11.72049312	8.61725575	9.88E-02	2.87E-02	11.94837868	0.43258161	0.0018775	2.87E-02
17	11.69412707	8.59983785	3.16E-02	9.08E-03	11.92201054	0.43258683	0.00187224	9.08E-03
18	11.68571741	8.59436551	1.00E-02	2.87E-03	11.91360021	0.43258849	0.00187051	2.87E-03
19	11.68304848	8.59264179	3.18E-03	9.09E-04	11.91093107	0.43258902	0.00186995	9.09E-04
20	11.68220328	8.59209796	1.01E-03	2.87E-04	11.9100858	0.43258919	0.00186977	2.87E-04
21	11.68193588	8.59192623	3.18E-04	9.08E-05	11.90981838	0.43258924	0.00186972	9.09E-05
22	11.68185132	8.59187198	1.00E-04	2.87E-05	11.90973381	0.43258925	0.0018697	2.87E-05
23	11.68182459	8.59185483	3.18E-05	9.08E-06	11.90970708	0.43258926	0.00186969	9.07E-06
24	11.68181614	8.59184941	1.00E-05	2.87E-06	11.90969863	0.43258926	0.00186969	2.86E-06
25	11.68181347	8.5918477	3.17E-06	9.08E-07	11.90969596	0.43258926	0.00186969	9.01E-07

ii) If Φ use $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}^2_2(\xi)$. On the other hand, from ***Table 6*** *and* ***Corollary 4.4****, $P_{f,13} \approx 2.02 \in B(p;\delta_K)$, which is consistent with* ***Corollary 2.1****, since in general $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, which follows from the proof of* ***Proposition 4.1****. So, it is concluded that Φ has an order of convergence (at least) quadratic, that is, $\Phi \in \mathrm{Ord}^2(\xi)$.*

Table 6: Iterations generated by the fractional quasi-Newton method accelerated.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2026.628123	2025.393793	1.38E+03	2.02E+03	2027.016086	0.03400122	9.21E-05	2.02E+03
2	1343.689808	1341.54807	9.66E+02	1.33E+03	1344.023514	0.16909715	0.0002274	1.33E+03
3	872.9738032	870.1756552	6.66E+02	8.62E+02	873.2701122	0.26221217	0.0004193	8.62E+02
4	554.9152667	551.6514102	4.50E+02	5.43E+02	555.1863071	0.32512915	0.00067737	5.43E+02
5	344.7596997	341.173768	2.97E+02	3.33E+02	345.014044	0.36670122	0.00099809	3.33E+02
6	209.3841806	205.5840939	1.92E+02	1.97E+02	209.6277698	0.39348063	0.0013556	1.97E+02
7	124.6869147	120.7550587	1.20E+02	1.12E+02	124.9237751	0.41023508	0.00170264	1.12E+02
8	73.46486534	69.46807813	7.25E+01	6.09E+01	73.69765627	0.4203676	0.00198789	6.09E+01
9	43.70741877	39.70792607	4.21E+01	3.11E+01	43.93784558	0.42625409	0.00217704	3.11E+01
10	27.24135725	23.30847331	2.32E+01	1.47E+01	27.47047589	0.42951134	0.00225857	1.47E+01
11	18.66834592	14.88366691	1.20E+01	6.33E+00	18.89678346	0.43120722	0.0022372	6.33E+00
12	11.68178703	8.59184524	9.40E+00	2.36E-05	11.90966952	0.43258927	0.00186968	2.36E-05
13	11.68181223	8.59184691	2.53E-05	4.23E-10	11.90969472	0.43258926	0.00186969	1.41E-08

From the previous results, it is observed that there exists a considerable improvement in the order of convergence between the matrices (72) and (75). Therefore, it may be established that it is more efficient to solve the function (66) by implementing the fractional quasi-Newton method accelerated in the function (69). So, by providing multiple values of the parameters DNI and T_{air}, it is possible to obtain a histogram of the efficiencies of a hybrid solar receiver analogous to the one shown in Figure 2. Finally, it is necessary to mention that the **Corollary 4.2** can also be implemented in the **generalized fractional quasi-Newton method**, which is obtained by using the matrix (72) with the following function

$$g_{a,b,f}(x) := af(x_i) + f^{(1)}(x_i)(x - bx_i), \quad a,b \in \mathbb{R}, \tag{76}$$

as a consequence, it is possible to define the following set of matrices

$$\left\{A_{g,\alpha} = A_{g,\alpha}\left(o_x^\alpha\right) \ : \ \exists A_{g,\alpha}^{-1} \in {}_n\mathrm{IM}_{x,\alpha}^{\infty}(g) \ \text{ and } \ o_x^\alpha \in {}_n\mathrm{O}_{x,\alpha}^1(g)\right\} \cap \left\{o_x^\alpha \ : \ o_k^\alpha c \neq 0 \ \forall c \in \mathbb{R}\setminus\{0\} \ \text{ and } \ \forall k \geq 1\right\}, \tag{77}$$

and therefore, it is possible to define the following sets of fractional iterative methods

$$\left\{\Phi \ : \ A_{g,\beta} \text{ uses } g = g_{a,0,f} \text{ with } a \in (0,1]\right\}, \tag{78}$$

$$\left\{\Phi \ : \ A_{g,\beta} \text{ uses } g = g_{1,b,f} \text{ with } b \in (0,1]\right\}, \tag{79}$$

which correspond to two uncountable families of fractional fixed-point methods in which the **Corollary 4.2** may be implemented. Finally, it is necessary to mention that fractional iterative methods may be defined in the complex space [18], that is,

$$\{\Phi(\alpha,x) \ : \ \alpha \in \mathbb{R}\setminus\mathbb{Z} \ \text{ and } \ x \in \mathbb{C}^n\}. \tag{80}$$

However, due to the part of the integral operator that fractional operators usually have, it may be considered that in the matrix $A_{g,\alpha}$ each fractional operator o_k^α is obtained for a real variable $[x]_k$, and if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]_k$, that is,

$$A_{g,\alpha}(x_i) := A_{g,\alpha}(x)\Big|_{x \longrightarrow x_i}, \quad x \in \mathbb{R}^n, \quad x_i \in \mathbb{C}^n. \tag{81}$$

Therefore, it is possible to obtain the following corollaries:

Corollary 5.1. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $f \in \mathrm{Ord}_{NR}^2(\xi)$, let $g : \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $g^{(1)}(x) = f^{(1)}(x) \ \forall x \in B(\xi;\delta)$, and let $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function given by the equation (44). So, for each operator $o_x^\alpha \in {}_n\mathrm{O}_{x,\alpha}^1(g)$ such that $A_\alpha\left(o_x^\alpha\right) \in {}_n\mathrm{G}_{FNR}(\alpha)$, there exists the matrix $A_{g,\alpha}^{-1} = A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(g)$ such that it fulfills the following condition*

$$\lim_{\alpha\to 1} A_{g,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \ \forall x \in B(\xi;\delta). \tag{82}$$

As a consequence, by the ***Corollary 4.2****, if* $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \Rightarrow \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.

Corollary 5.2. *Let* $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ *be a function such that* $f \in \mathrm{Ord}_{NR}^2(\xi)$*, let* $g : \mathbb{C}^n \to \mathbb{C}^n$ *be a function such that* $g^{(1)}(x) = f^{(1)}(x) \ \forall x \in B(\xi;\delta)$*, and let* $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ *be an iteration function given by the equation* (44)*. So, for each finite sequence of operators* $\left\{o_{k,x}^\alpha\right\}_{k=1}^N \in {}_n\mathrm{O}_{x,\alpha}^1(g)$ *such that it fulfills the following conditions*

$$\lim_{\alpha\to 1}\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) = N\nabla_x \quad \textit{and} \quad A_\alpha\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) \in {}_n\mathrm{G}_{FNR}(\alpha), \tag{83}$$

where ∇_x *denotes the gradient operator. It is possible to construct the following matrix*

$$A_{g,\alpha}^{-1} = \frac{1}{N} A_\alpha\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) \circ A_\alpha^T(g). \tag{84}$$

As a consequence, by the ***Corollary 4.2****, if* $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \Rightarrow \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.

Corollary 5.3. *Let* $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ *be a function such that* $f \in \mathrm{Ord}_{NR}^2(\xi)$*, let* $\{g_k\}_{k=1}^N$ *be a finite sequence of functions* $g_k : \mathbb{C}^n \to \mathbb{C}^n$ *such that it fulfills the following condition*

$$\left(g_1^{(1)} + g_2^{(1)} + \cdots + g_N^{(1)}\right)(x) = N f^{(1)}(x) \ \forall x \in B(\xi;\delta), \tag{85}$$

and let $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ *be an iteration function given by the equation* (44)*. So, for each fractional operator* o_x^α *that fulfills the following conditions*

$$o_x^\alpha \in \bigcap_{k=1}^N {}_n\mathrm{O}_{x,\alpha}^1(g_k) \quad \textit{and} \quad A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(g) \in {}_n\mathrm{IM}_{x,\alpha}^\infty(g), \tag{86}$$

where $g = g_1 + g_2 + \cdots + g_N$*. It is possible to construct the following matrix*

$$A_{g,\alpha}^{-1} = \frac{1}{N} A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(g_1 + g_2 + \cdots + g_N). \tag{87}$$

As a consequence, by the ***Corollary 4.2****, if* $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \Rightarrow \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.

Corollary 5.4. *Let* $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ *be a function such that* $f \in \mathrm{Ord}_{NR}^2(\xi)$*, let* $\{g_k\}_{k=1}^N$ *be a finite sequence of functions* $g_k : \mathbb{C}^n \to \mathbb{C}^n$ *such that it defines a finite sequence of operators* $\left\{o_{k,x}^\alpha\right\}_{k=1}^N$ *through the following condition*

$$o_{k,x}^\alpha \in {}_n\mathrm{MO}_{x,\alpha}^{\infty,u}(g_k) \ \forall k \geq 1, \tag{88}$$

and let $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ *be an iteration function given by the equation* (44)*. So, if there exists a matrix* $A_{N,\alpha}$ *such that it fulfills the following conditions*

$$\exists A_{N,\alpha}^{-1} = \sum_{k=1}^N A_\alpha\left(o_{k,x}^\alpha\right) \circ A_\alpha^T(g_k) \quad \textit{and} \quad \lim_{\alpha\to 1} A_{N,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \ \forall x \in B(\xi;\delta). \tag{89}$$

As a consequence, by the ***Corollary 4.2****, if* $\Phi\left(A_{N,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \Rightarrow \Phi\left(A_{N,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.

6. Conclusions

In all the examples shown, a decrease in the number of iterations necessary to converge to the solutions is observed when implementing the function (59) in the fractional quasi-Newton method, which means that the generated sequences show an acceleration in their order of convergence, which was to be expected given the **Corollary 4.2**. The fractional fixed-point methods, such as the fractional Newton-Raphson method, can find multiple zeros of a function using a single initial condition, this partially solves the intrinsic problem of classical iterative methods, which is that in general, to find N zeros of a function, N initial conditions must be provided. Due to the fractional operators implemented, these methods may be considered **non-local parametric iterative methods**, so they have two important characteristics [18]:

i) The initial condition does not necessarily have to be close to the sought values due to the non-local nature of fractional operators .
ii) When working in a space of N dimensions, in the case that it is necessary to change the initial condition, unlike the classical iterative methods where, in the worst case, it is necessary to vary the N components of the initial condition until a suitable value is obtained, in the fractional fixed-point methods, it is enough to vary the parameter α of the fractional operators until an adequate value is found that allows generating a sequence that converges to a sought value.

The above features, make the fractional fixed-point methods an ideal numerical tool for working with nonlinear algebraic equation systems that vary with time-dependent parameters, as is the case of the functions (66) and (69), which allows studying the behavior of temperatures and efficiencies of a hybrid solar receiver [25, 26]. Due to many nonlinear algebraic systems related to engineering and physics are often related to time-dependent parameters. Having a way to classify and accelerate the order of convergence of fractional fixed-point methods through the **Corollary 4.2**, may become a fundamental piece to continue expanding the applications of fractional operators.

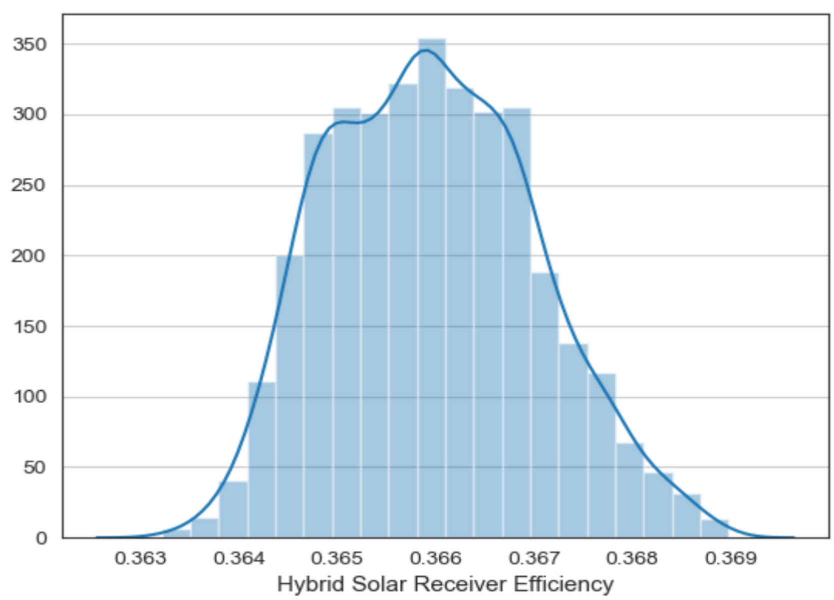


Figure 2: Histogram and density curve of the efficiency of a hybrid solar receiver obtained from a simulation corresponding to a period of thirty days, which is equivalent to 2410 pairs of parameters (DNI, T_{air}) randomly generated on the domain $[12, 958] \times [11, 45]$. The selected domain is based on data measured in real-time at the Center for Advanced Studies in Energy and Environment (CEAEMA) [25, 37]. The values generated for the simulation presented the mean values $mean(DNI) = 662.35$ and $mean(T_{air}) = 31.28$, with sample standard deviations $std(DNI) = 257.83$ and $std(T_{air}) = 6.11$, while the values of the efficiencies were obtained through the solutions of the function (69) using the fractional quasi-Newton method accelerated.

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