Median Problems in Networks

Vladimir Marianov[1]
Department of Electrical Engineering
Pontificia Universidad Católica de Chile, Santiago, CHILE
e-mail: marianov@ing.puc.cl

and

Daniel Serra
Department of Economics and Business
Universitat Pompeu Fabra, Barcelona, SPAIN
e-mail: daniel.serra@upf.edu

Abstract

The P-median problem is a classical location model "par excellence". In this paper we, first examine the early origins of the problem, formulated independently by Louis Hakimi and Charles ReVelle, two of the fathers of the burgeoning multidisciplinary field of research known today as Facility Location Theory and Modelling. We then examine some of the traditional heuristic and exact methods developed to solve the problem. In the third section we analyze the impact of the model in the field. We end the paper by proposing new lines of research related to such a classical problem.

[1] This research has been possible thanks to a grant by the Spanish Ministry of Science and Education, SEJ2006-12291

Introduction: Early History, brief review of p-median, its uses and chapter outline

The p-median problem is, with no doubt, one of the most studied facility location models. Basically, the p-median problem seeks the location of a given number of facilities so as to minimize some measure of transportation costs, such as distance or travel time. Therefore, demand is assigned to the closest facility.

The p-median problem is widely used in both public and private sector location decisions. Its uses include the original practical case suggested by Hakimi, which is locating a number of switching centers on a telephone network, as well as a large number of other applications, both geographical and not-geographical. Among the first, it is worth mentioning the location of public facilities so that the distance the public must travel to them is minimized: schools and hospitals are a typical example. Non-geographical applications arise for example when there is the need of grouping or clustering objects, tasks, events, and so on.

Why is it called p-*median*? The *median vertex* is the vertex of a networ

k or graph for which the sum of the lengths of the shortest paths to all other vertices is the smallest. Locating a school on the median vertex of a network in which edges or arcs represent roads and each node represents a child, minimizes the total distance that children have to walk to go to that school. Or, if each node represents a customer, and a maintenance center housing a vehicle has to be located on some vertex of the network, the median vertex will be the location that minimizes the total distance traveled by the vehicle, if all customers have to be served, one at a time.

On a network, finding the median vertex solves a problem similar to that posed by Fermat on a (Euclidean) plane in the 1600's, consisting of finding the location of the point on a plane which minimizes the sum of its distances to three points whose location is known. Weber, in the early 20th century, generalized this problem by adding weights to them, which could represent amount of demand or population aggregated at the points. If a facility is located at this weighted median, it will satisfy the demand of the three points with the minimal transportation cost. Later, the Weber problem was generalized to include more than three demand points, and to locate more than one facility. The version with multiple facilities became known as the Multi-Weber problem. In the 20th century, Cooper (1963, 1964) provided heuristic solutions for it.

Although now it seems a natural step, Hakimi did not formulate the *p*-median as an integer programming problem. This was first done by ReVelle and Swain (1970) who, not being familiar with the results of Hakimi, assumed node-only location of what they called central facilities. This formulation opened a new line in the search of solution procedures for the *p*-median problem.

The *p*-median can be formally stated in words as:

> "Given the location of *n* points that house known amounts of demand, designate *p* of these points as facilities and allocate each demand to a facility, in such a way as to minimize the total weighted distance between demands and facilities"

This problem can be solved using different methods. Total enumeration is always an alternative, although its complexity makes this method useless when the problem grows. The first methods that were proposed for solving the p-median were heuristic. Among these, Maranzana (1964) describes a heuristic that randomly locates the *p* facilities and then solves the allocation problem (which has a polynomial complexity). Each facility in this initial solution serves a set or cluster of demands. Once this solution is found, Maranzana iteratively relocates the facilities within each cluster if it improves the solution, and reallocates demands keeping fixed the locations of the facilities, which potentially changes the clusters. A stable solution is reached, which is the best, but not necessarily optimal. Teitz and Bart (1968) proposed a method called "vertex substitution", that, starting from a known solution, relocates facilities one by one (and reallocates demands), whenever this relocation improves the solution. When no more improvements are possible by this method, a good solution has been reached. These heuristics are studied in detail in a different chapter of this book.

The works of Maranzana (1964) and Teitz and Bart (1968) were known when ReVelle and Swain (1970) proposed an optimal procedure for the *p*-median, based on linear programming and branch and bound. Their formulation is now well known and used profusely, in a slightly different form:

p-median:

$$\text{Min} \sum_{i,j} h_i d_{ij} x_{ij} \quad (1)$$

s.t.

$$\sum_j x_{ij} = 1 \qquad i = 1, 2, \ldots n \quad (2)$$

$$x_{ij} \le y_j \qquad i, j = 1, 2, \ldots n. \quad (3)$$

$$\sum_j y_j = p \quad (4)$$

$$x_{ij}, y_j \in \{0,1\} \qquad i, j = 1, 2, \ldots n. \quad (5)$$

Where:

i	Index of demand points
m	Total number of demand points in the space of interest
j	Index of potential facility sites
n	Total number of potential facility locations
h_i	Weight associated to each demand point.
d_{ij}	Distance between demand area *i* and potential facility at *j*.
x_{ij}	Variable that is equal to 1 if demand area *i* is assigned to a facility at *j*, and 0 otherwise
y_j	Variable that is equal to 1 if there is an open facility at *j*, and 0 otherwise.

The first set of constraints forces each demand point to be assigned to only one facility. The second set of constraints allows demand point *i* to assign to a point *j* only if there is an open facility in this location. Finally, the last constraint sets the number of facilities to be located.

The second set of constraints is known as the "Balinski" constraints (Balinski, 1965), since he was the one to write them in this form in 1965, when studying the Simple Plant Location Problem. An alternative condensed version of the problem can be formulating by substituting the "Balinski" constraints with the following set:

$$\sum_{j=1}^{n} x_{ij} \leq m y_j \qquad i = 1,2,\cdots,m \qquad (6)$$

This constraint states that no demand node can assign to point *j*, unless there is a facility open there. While this set of constraints substantially reduces the size of the problem, when solving it using linear programming without any integer requirements will nearly produce all x_{jj} fractional. On the other hand, the "Balinski" set of constraints makes the problem at hand quite large as the number of constraints required together with the number of variables are very large even in relatively small problems. Nevertheless, when solving the P-Median Problem in its extended form using linear programming relaxation most solutions are integer. ReVelle and Swain (1970) observed that when branch-and-bound was required to resolve fractional variables produced by linear programming, the extent of branching and bounding needed was very small, always less than 6 nodes of a branch-and-bound tree. Therefore, the expanded form of the constraint makes integer solutions far more likely. Infact, in this formulation, only the location variables y_j need to be declared binary, as ReVelle and Swain (1970) proved. Morris (1978) , solved 600 randomly generated problems of the very similar Simple Plant Location Problem with the extended form of the constraint and found that only 4% did require the use of branch-and-bound to obtain integer solutions. Rosing et al. (1979c) proposed several ways to reduce both the number of variables and constraints in order to make the P-Median Problem more tractable.

Since these early contributions, many methods have been proposed for solving this problem, as well as variations of the problem that consider additional constraints, cost functions and assignment policies.

Louis Hakimi was one of the first researchers addressing the problem on a network. In his 1964 paper, the best location of a facility was sought, considering that all demand must be attended. Similarly to the problem on a plane, the demand is distributed over the region of interest. In the network version of the problem, demand is located only on vertices or nodes, each of them having a weight representing the total amount of demand that it houses. In Hakimi's version, the facility can be located on a node or at a point on an edge of the network, distinction that does not exist when the problem lies on the plane. Hakimi proved, however, that there is always an optimal solution at a node. The problem consists in finding this optimal location, in such a way that the sum of the distances between the facility and each demand node, weighted by the amount of demand, is minimum. Because of this minimization of a sum of terms, the problem has also been called "minsum", or "minisum" problem.

In 1965, Hakimi was able to generalize his main result (node solution) to the case of multiple facilities. Now, the problem consists of finding the locations of *p* facilities, in such a way that the sum of the weighted distances between each demand node and its closest facility is the least. He called this problem the *p*-median. Note that the presence of more than one facility introduces an additional level of difficulty, since the solution must now answer to two questions: where to locate the *p* facilities – the "location" problem; and what demand node is assigned to which facility –the "allocation" problem. In the Hakimi (1965) version, the allocation problem is defined as assignment of demand nodes to their closest facilities. However, the location of multiple facilities allows different possibilities, including allocation of a demand to more than one facility, which could be optimal if facilities have a limited capacity, or if customers located at demand nodes can choose different facilities in different opportunities.

In this chapter, we first review and synthesize the early contributions by Hakimi (1964) and (1965), as well as ReVelle and Swain (1970), in separate sections. A further section is devoted to assess the impact that these works had on the discipline. Then, we review the major contributions that followed the first papers, and propose some open questions related to the *p*-median problem. We end with conclusions.

Hakimi 1964: synthesis

The concept of the *median vertex* of a graph, as well as some methods for finding the solution for the Multi-Weber problem (including the case with node weights, representing the amounts of demand at the nodes), were known when Hakimi (1964) posed the problem of finding the "absolute median" of a graph. The absolute median was a generalization of the median, in which the facility can be located not only on nodes, but also at any point along an edge of the network. This generalization is possible only on a network.

Hakimi's (1964) paper not only generalized the concept of median vertex, but also did the same to the *center vertex*, which is the vertex whose maximum distance to any other node of the network is minimized. He defines the "absolute center", which is located anywhere on the network. The center problem is addressed in a different chapter of this book.

The application in which Hakimi was interested was that of locating a telephone switching center (or switch), *S*, in a communication network. He represents this communications system as a finite graph or network *G*. In such a graph, the switching center is directly connected through wires to each vertex v_i. Any message or communication between two vertices must be established through this switch. Each vertex v_i –connected through a branch b_i to the switch *S*, could need more than a pair of wires to evacuate its traffic[2]. The number of wires needed by vertex v_i (its weight) is h_i, and the cost or length of the branch b_i is w_i. Such a network has the shape of a star, having the switch on its center. The problem is to find the optimal location of the switching center in such a way that the total length of the wires is minimal.

Hakimi first remarks that the usual concept of median vertex does not apply to this problem, since the switch *S* could be located anywhere on the network, including both vertices and branches or edges. Then, he defines the distance $d(x,y)$ on the network or graph between points *x* and *y* on the network, as

[2] In a telephone network, this number of wires is associated to the number of subscribers at vertex v_i. In other communications networks it could represent the (discrete) capacity of the branch.

the length of the shortest path between *x* and *y*, where the length of a path is the sum of the weights of the branches on that path; i.e., the sum of the length of the segment of branch connecting the point *x* to the switch, multiplied by the weight of the branch (weighted length), plus the length of the segment of branch connecting *y* to the switch ***S***, times the weight of that branch. If both points are on the same branch, it is just the length of the segment connecting them, times its weight.

Using this notation, the point y_0 on an element of a weighted *n*-vertex graph *G* is defined as the *absolute median* of *G* if the sum of the weighted shortest distances between y_0 and every point *y* on G

$$\sum_{i=1}^{n} h_i d(v_i, y_0) \leq \sum_{i=1}^{n} h_i d(v_i, y) \qquad (6)$$

This point is identified with the optimal location of the switch in the communications network.

After defining the absolute median, Hakimi proposes a method for finding the location of the median vertex: this is done by writing the $n \times n$ distance matrix of the graph, adding up all the elements of each column *j* (distances between the node *j* and the remaining nodes) and choosing as the median the node *j* corresponding to the column with the least value of the sum of distances.

Hakimi then states the main median-related result of the paper –a theorem, which he recognizes being the generalization of an unpublished result communicated to him by A. J. Goldstein, at Bell Labs. Goldstein proved that an absolute median of a tree is always located in a vertex.

Theorem (Hakimi 1964): An absolute median of a graph is always at a vertex of the graph.

Hakimi proves that, if x_0 is an arbitrary point on the graph, not on a vertex, there always exist a vertex v_m of *G* such that

$$\sum_{i=1}^{n} h_i d(v_i, v_m) \leq \sum_{i=1}^{n} h_i d(v_i, x_0) \qquad (7)$$

i.e. there is an absolute median at a vertex v_m. Note that this result does not preclude other absolute medians existing on the network.

Rather than repeating Hakimi's proof, we repeat his reasoning, reducing the mathematics as possible. The point x_0 is assumed to be located on an edge (v_a, v_b). Assume also that nodes are re-indexed in such a way that the following is true: the point x_0 is now located on the edge (v_p, v_{p+1}), and for all nodes with indices smaller than or equal to p, the shortest path connecting the node and point x_0 goes through node v_p, (i.e. connected through the left of x_0) while for all nodes with indices larger than p, the shortest path between the node and point x_0 goes through node v_{p+1} (i.e. through the right of x_0). The total weighted distance can then be expressed as the sum of two terms, representing the sum of the weighted distances to the left-side nodes, and to the right-side nodes, respectively:

$$\sum_{i=1}^{n} h_i d(v_i, x_0) = \sum_{i=1}^{p} h_i d(v_i, x_0) + \sum_{i=p+1}^{n} h_i d(v_i, x_0)$$

In turn, each distance can be decomposed in two as follows:

$$\sum_{i=1}^{n} h_i d(v_i, x_0) = \sum_{i=1}^{p} h_i d(v_i, v_p) + \sum_{i=1}^{p} h_i d(v_p, x_0) + \sum_{i=p+1}^{n} h_i d(v_i, v_{p+1}) + \sum_{i=p+1}^{n} h_i d(v_{p+1}, x_0) .$$

Since $\sum_{i=p+1}^{n} h_i d(v_{p+1}, x_0) = \sum_{i=p+1}^{n} h_i d(v_{p+1}, v_p) - \sum_{i=p+1}^{n} h_i d(v_p, x_0)$

The full expression is

$$\sum_{i=1}^{n} h_i d(v_i, x_0) =$$

$$\sum_{i=1}^{p} h_i d(v_i, v_p) + \sum_{i=1}^{p} h_i d(v_p, x_0) + \sum_{i=p+1}^{n} h_i d(v_i, v_{p+1}) + \sum_{i=p+1}^{n} h_i d(v_{p+1}, v_p) - \sum_{i=p+1}^{n} h_i d(v_p, x_0)$$

$$= \sum_{i=1}^{p} h_i d(v_i, v_p) + \sum_{i=p+1}^{n} h_i d(v_i, v_{p+1}) + \sum_{i=p+1}^{n} h_i d(v_{p+1}, v_p) + \left[\sum_{i=1}^{p} h_i - \sum_{i=p+1}^{n} h_i \right] d(v_p, x_0).$$

The term in square brackets is the sum of node weights "on the left", minus the sum of the node weights "on the right" of the point x_0. Without loss of generality, suppose that the sum of node weights on the left is larger than or equal to the sum on the right. Then, the term in square brackets is non-negative, and by reducing the distance $d(v_p, x_0)$ that multiplies the square bracketed term, i.e. moving the point x_0 to the left, the total sum is reduced or, at most, stays the same. The minimum value for this distance is zero, which happens when the median point x_0 is located on the node v_p.

The same argument can be repeated when the sum of the node weights on the right of x_0 is strictly larger than the sum of node weights on the left. In that case, the term in square brackets is strictly negative, and moving the point x_0 to the right strictly reduces the value of the total sum. The best value is obtained when x_0 is located on top of v_{p+1}.

This proves that there is always a median point on a vertex of the graph, either on the left or the right of a point x_0 on an edge, i.e., for any point x_0,

$$\sum_{i=1}^{n} h_i d(v_i, v_m) \leq \sum_{i=1}^{n} h_i d(v_i, x_0),$$

and, although an absolute median can be defined, it is always a vertex median.

Hakimi concludes that the median is the best location for a switch in a communications network. It also could be a good location for a police station if h_i is the average number of daily automobile accidents in community *i*, and the police must visit the scene of each accident to make a report. He also suggests a mixed approach, in which a combination between the median and the center points is sought.

Hakimi 1965: synthesis

The single median problem answers the question of the optimal location of a single facility. When more than a facility is to be located (say *p* facilities), the problem becomes known as the "*p*-median" problem, term that was first used by Hakimi (1965) in the sequel of his 1964 paper. As before, Hakimi studies the *p*-median as a model that solves the problem of locating *p* switching centers on a communications network. Also in this paper, he studies a related problem, which we now know as the Location Set Coverage Problem (Toregas et al. 1971), applied to finding the least number of policemen to be deployed on a highway network, in such a way that nobody is farther away from a policeman that a preset distance. Although the paper breaks ground for two of the most well-known models, we concentrate on the *p*-median.

When two or more facilities need to be located, on the plane or on a network, there is an extra degree of difficulty: the allocation or assignment of demands to facilities. In other words, the decision regarding what facility or facilities will serve the demand at each node. In Hakimi (1965), demands are assigned to their closest facilities.

In this paper, Hakimi generalizes once more the definition of the median of a graph. If X_p is a set of *p* points x_1, x_2, … x_p, and the distance of a node v_i to X_p is:

$$d(v_i, X_p) = \min\left[d(v_i, x_1), d(v_i, x_2), ..., d(v_i, x_p) \right] ,$$

i.e. the distance between the node v_i and its closest point x_k in X_p, then the set X_p^* is a "*p*-median" of the graph *G*, if for every X_p on *G*,

$$\sum_{i=1}^{n} h_i d(v_i, X_P^*) \leq \sum_{i=1}^{n} h_i d(v_i, X_P) .$$

In other words, X_p^* is the set of *p* points on the graph such that, if these points were facilities of some sort, the total weighted distance between the demands and their closest facility would be minimized. Hakimi identifies the *p*-median with the optimum locations of p switching centers in a communications network.

He then derives the main result of the paper, which is to extend the validity of the all-node solution to the *p*-median case.

Theorem (Hakimi (1965): There exists a subset V_p^* of the set of vertices, containing *p* vertices such that for every set of p points *X* on *G*

$$\sum_{i=1}^{n} h_i d(v_i, V_P^*) \leq \sum_{i=1}^{n} h_i d(v_i, X)$$

For the proof, Hakimi assumes that the allocation problem has been solved; i.e., there are *p* clusters of demand points, each cluster *j* consisting of a point x_j in *X* and a set of demands for which x_j is the closest point in *X*. Then, if the point x_j is on an edge, by the Theorem (Hakimi 1964), there is always a vertex v^*_j such that

$$\sum_{i \in cluster\ j} h_i d(v_i, v_j^*) \le \sum_{i \in cluster\ j} h_i d(v_i, x_j)$$

The same inequality can be derived for each cluster. Note that the allocation of demands has not changed; the demands that were in cluster *j* are still in the same cluster. Adding up all these inequalities, Hakimi obtains:

$$\sum_{j} \sum_{i \in cluster\ j} h_i d(v_i, v_j^*) \le \sum_{i=1}^{n} h_i d(v_i, X)$$

The left hand side of this inequality consists of the sum of *p* terms, one for each one of the original clusters. However, as the median point in each cluster moves toward a vertex, it might happen that a demand node becomes reassigned to a different node in V_p^*. This only happens if the re-allocation contributes to decrease still more the total sum, so

$$\sum_{i=1}^{n} h_i d(v_i, V_j^*) \le \sum_{j} \sum_{i \in cluster\ j} h_i d(v_i, v_j^*),$$

and

$$\sum_{i=1}^{n} h_i d(v_i, V_j^*) \le \sum_{i=1}^{n} h_i d(v_i, X)$$

After stating this Theorem, Hakimi describes a method to find the *p*-median of a graph, consisting basically in enumerating all possibilities. As an example, he computes the 3-median of a 10-node graph.

ReVelle and Swain 1970: synthesis

ReVelle and Swain (1970) addressed the problem they call "central facility location", consisting of designating *m* of *n* communities in a geographical region as centers, so that the average time or distance travelled by people to go to these centers is minimum. They also suggest that the formulation they use is applicable to the case in which the facilities are supply points from where good emanate to the communities.

The average distance is

$$\bar{d} = \frac{\sum_{i=1}^{n} h_i d(v_i, V_m^*)}{\sum_{i=1}^{n} h_i}$$

where $d(v_i, V_m^*)$ is the distance between the community (demand node) *i*, and its closest center, belonging to the set of centers V_m^*.

Note that ReVelle and Swain, coming from a different discipline that Hakimi, rather than using the graph-theoretical notation (median), use a term common among geographers (center) for the same concept. They also include references to previous works in the incipient area of discrete location analysis, most of them on the Simple Plant Location Problem, also called Uncapacitated Facility Location Problem (UFLP) or Simple Plant Location Problem (SPLP). Of particular interest is the reference to the work of Efroymson and Ray (1966), who used the Land and Doig (1960) method, which was later called Branch and Bound, applied to a new formulation of the SPLP. Hakimi's papers are not among the references.

ReVelle and Swain remark that they use linear programming tools for solving the central facility location (*p*-median) problem and that, in the unlikely event of a non-integer solution, a branch-and-bound scheme is recommended. They also suggest that a heuristic solution can be tested for optimality by using linear programming, or can be used as a good starting guess for the optimal solution.

The assumptions are similar to those of Hakimi: travel is performed using the shortest path between a community and a center; allocation cannot be partial; i.e., a community (or demand node) is assigned fully to one and only one center (which later they prove that is an optimal choice, provided that communities with a center allocate to themselves). An extra assumption is that all centers are located at communities, and there are no candidate locations other than communities.

Once the matrix of shortest distances $d(v_i, v_j)$ between communities (vertices) v_i and v_j is computed for all i and j, and the allocation variables are defined as

$$x_{ij} = \begin{cases} 1 \text{ ifcommunity } i \text{ is assigned to center } j \\ 0 \text{ otherwise} \end{cases},$$

the p-median or center facility location problem can be formulated as a linear programming problem:

$$\text{Min } \sum_{i,j} h_i d(v_i, v_j) x_{ij}$$

s.t.

$$\sum_j x_{ij} = 1 \qquad i = 1, 2, \dots n$$

$$x_{ij} + \sum_{k \neq j} x_{jk} \leq 1 \qquad i, j = 1, 2, \dots n, i \neq j.$$

The last constraint requires that, if community i assigns to community j, the last one must be assigned to itself. Note that if a community j assigns to itself ($x_{jj} = 1$), then the community must house a facility or center. The authors explain that this constraint can be replaced by the simpler one (suggested by Prof. Falkson, at Cornell):

$$x_{ij} \leq x_{jj} \qquad i, j = 1, 2, \dots n.$$

Finally, a constraint is added to enforce the required number of facilities (p):

$$\sum_j x_{jj} = p$$

A proof is offered next, of the fact that, if all self-assignments (x_{jj}) are either zero or one, then there is an optimal solution that considers each of the communities assigned fully to one facility (variables x_{ij}

are either zero or one). Basically, ReVelle and Swain argue that, if a community's demand was divided among two or more facilities in the solution, this solution could not be optimal. In effect, unless the community is equidistant from these two or more facilities, that proportion of the demand assigned to the farthest facility can be reassigned to the closest, and the objective would decrease, indicating that the solution is not optimal. If a community is equidistant from two or more facilities, the solution is an alternate optimum, and can be substituted by a solution with full assignment (binary variables x_{ij}.)

In order to solve the problem, linear programming is recommended. In the event that there appears a fractional assignment, branch and bound is used on the x_{jj} variables, and the variable on which to branch is chosen by a rule that considers to branch first on the variable x_{kk} for which the term $\left(\min_{j \neq k}\left\{h_k d(v_k, v_j)\right\}\right)$ is the largest and variable x_{kk} has not been branched on.

The authors argue that the number of iterations needed to solve the problem using branch and bound may be favorably compared to enumeration, since the number of allocations that needs to be evaluated by enumeration is $\binom{n}{m}$, while the number of iterations in the branch and bound scheme was estimated by the authors in this case to be around $2(n^2 + 1)$. Since problems can grow large, the authors propose a way to cut down constraints, which consists in relaxing the constraints of the type

$$x_{ij} \leq x_{jj} \qquad i, j = 1, 2, \ldots n.$$

and solving the following problem:

$$\text{Min} \sum_{i,j} h_i d(v_i, v_j) x_{ij}$$

s.t.

$$\sum_j x_{ij} = 1 \qquad i = 1, 2, \ldots n$$

$$\sum_j x_{jj} = p$$

This problem can be solved to optimality just by inspection:

1. Assign every community to its closest neighbor, without allowing self-assignments.
2. Break the assignments of the *p* communities with the largest assignment costs and assign them to themselves.

Once obtained this solution, there will be some communities that receive assignments without being self- assigned. For these cases, add the corresponding constraints $x_{ij} \le x_{jj}$, and solve, now using linear programming and branch and bound.

The authors also suggest starting form a solution obtained by heuristic methods, although they do not propose any heuristic in particular. Regarding location of centers at places that do not correspond to communities, ReVelle and Swain suggest to add possible locations by adding the corresponding variables, and leaving population or demand zero.

The paper includes a discussion about the specification of the number of centers to locate, which the authors leave in the political arena. However, they still analyze the case in which a given maximum amount of funds is available. The total cost of a facility is given by

$$L_j = b_j x_{jj} + c_j \sum_i a_i x_{ij}$$

where b_j is the fixed cost and c_j the unitary expansion cost, which is multiplied by the amount of demand assigned to the center, i.e. its needed size. The total cost is the sum of the costs of all the facilities, and this total cost must not exceed the amount of funds, *M*.

$$\sum_j b_j x_{jj} + \sum_j c_j \sum_i a_i x_{ij} \le M$$

If both the fixed cost of establishing a center and the expansion cost of an already located facility are the same, independently of the site of the facility, then

$$b \sum_j x_{jj} + c \sum_i a_i \sum_j x_{ij} \le M$$

or

$$\sum_j x_{jj} \leq \frac{M - c\sum_i a_i}{b}$$

i.e.

$$p \leq \left\lfloor \frac{M - c\sum_i a_i}{b} \right\rfloor$$

After this analysis, the authors suggest solving the problem with different values of *p*, which provides insight on the trade-off between the travel time (or distance) and the number of centers.

ReVelle and Swain report computational experience for a problem with 10 communities and four facilities; they also report a short execution time (considering the available computing power in 1970) for a 30-community, 6-node problem.

Finally, ReVelle and Swain elaborate on the conditions in which a problem may have non-integer solutions. They identify the appearance of fractional optimal solutions with matrices that have what they call "counter-cycles" of costs, i.e., cycles of costs running in opposite directions (a fact that is related to non-symmetric matrices). However, they were not able to find a general rule for what was the condition that makes these fractional solutions appear; they even show an example of a matrix with counter-cycles that results in an all-integer solution for the *p*-median problem.

To close the paper, a paragraph is included on applications: clinics providing therapy for individuals with chronic diseases, warehouses, mail-sorting facilities, central schools and parks, and others.

Major impact of the early contributions (Hakimi 64, 65; ReVelle and Swain 70)

A common opinion among location researchers is that the paper by Hakimi's 1964 strongly contributed to trigger the interest in location theory and analysis, and started a long string of related publications, which does not seem to be decreasing. This opinion is somehow confirmed by the increasing yearly frequency of papers on location since 1964 (Tansel et al 1983a). Even if this was not so, it can be safely said that, at least, Hakimi (1964, 1965) brought awareness to the *p*-median problem. Other pioneering works are due to Kuehn and Hamburguer (1963), who addressed the heuristic solution of the Simple plant Location Problem, and Maranzana (1964), who proposed a solution method for locating *m* supply points among *n* demand points, basically the same as the alternate location-allocation algorithm that Cooper (1964) uses as a starting solution for solving the continuous version of the problem.

Hakimi first generalized the problem of finding the median of a graph, as known up to the date of the publication of his papers, by defining the generalized median. This concept of a median located anywhere on the network, at least when there is a single median to be found, proved later (in the same paper of 1964) not to be too useful, since there is always an optimal location of the facility at a node. However, if the problem changes, as for example in the general absolute median problem of Minieka (1977), or the gravity p-median of Drezner and Drezner (2007), this all-node solution does not necessarily hold, and here is when Hakimi's visionary definition becomes very important. It neither holds in the case of the center point, addressed in the same Hakimi (1964) paper.

The property he proved (or Hakimi property), of existence of optimal solutions on vertices, allowed looking for the optimal solution of the problem over a finite set (the nodes), instead of having to search over an infinite and continuous set (anywhere on the network). This by itself is a huge leap.

For finding a solution, Hakimi used an enumeration method, for which the number of mathematical operations grow tremendously with the size of the instance of the problem. Other than enumeration, the first optimal method for finding a solution to a reasonably sized instance of the *p*-median problem was proposed by ReVelle and Swain (1970). Although they simply state the problem not allowing location on edges of the network, thanks to the Hakimi property, the ReVelle and Swain formulation solves the general problem optimally, using location and allocation variables, both binary, and a branch-and-bound procedure, that had been then recently proposed by Land and Doig (1960).

ReVelle and Swain contribution was not only the integer programming formulation, but also the application to the location of central facilities, by relating the *p*-median problem to that solved by Weber (Hakimi does not refer to the problem on the plane). Many applications of the *p*-median were found in the public sector, after ReVelle and Swain (1970).

A review of methods used for solving the *p*-median is offered in a different chapter of this book.

Related major works, prequels and sequels

Credit for pioneering work on the *p*-median must be given to Hua Lo-Keng and others (1962) who proposed an algorithm for locating the 1-median on trees (and networks with cycles), and proved that locating medians points on vertices was better than locating them somewhere along the edges. Their paper was intended for practitioners who need to set up threshing floors for dispersed wheat fields, so it did not include much mathematical insight. Their work was apparently not known in the western world until much later, since Hakimi did not reference it in 1964, and both Goldman (1971) and Kariv and Hakimi (1979) rediscovered the same algorithm several years later.

An also frequently forgotten contribution is that by Gülicher (1965) who found the same results as Hakimi, but was known (and it is currently known) better among economists, rather than being a reference among location scientists.

There are several major works presenting generalizations to the original *p*-median problem. Goldman (1969) generalized the *p*-median, defining a problem in which commodities are transported over a path between origin and destination nodes, and the total transportation cost is minimized. The path goes through one or two medians. Goldman generalized Hakimi´s property to this case. The problem addressed by Goldman (1969) has been extensely studied, and it is now known as the *p*-hub median location problem. In turn, this result was generalized by Hakimi and Maheshwari (1972) to multiple commodities and multiple intermediate medians case.

Holmes et al. (1972) introduced two interesting generalizations of the *p*-median. The first generalization is elasticity of the demand, i.e. a situation in which customers lose interest in the service or goods if these are located beyond a threshold distance. This generalization is useful in the case of non-essential goods or services. The second generalization considers a constraint on the capacity of the facilities. This constraint makes the problem much harder, since it leads to the appearance of many fractional-valued location and allocation variables in the solution, if the integer-programming problem is solved in a linearly relaxed version.

An interesting extension of the *p*-median model was its application to hierarchical systems, i.e., systems composed by more than one category of facilities. Calvo and Marks (1973) appear to be the first in exploring this type of setting. Their resulting model has multiple objectives. Further work with hierarchical systems was performed by Narula (1984).

Probabilistic behavior has also been included in the *p*-median models: Frank (1966), in an early sequel of Hakimi's contributions, discusses the effect of probabilistic demands. Mirchandani and Odoni (1979) extend the Hakimi result to the case in which the travel distances or times are random variables. Drezner (1987), on the other hand, addresses the "unreliable *p*-median" in which a facility has a certain probability of becoming inactive, and offers a heuristic for solving this problem. Finally, Berman and Larson (1982) formulate the Stochastic Queue Median, which locates a single facility operating as a M/G/1 queue, on any point of a network.

Wesolowsky and Truscott (1975) introduce the multiperiod *p*-median problem, in which the facilities are relocated in response to predicted changes in demand, considering that relocating facilities has a cost. A loosely related problem is solved by Serra and Marianov (1998), who determine the best locations for *p* facilities when the demand changes through the day.

Very relevant is the analysis of the properties and applicability of the *p*-median problem. A first contribution was that of Hillsman and Rhoda, 1978, who studied the effects of data aggregation in the *p*-median, i.e., of the fact that customers are considered as concentrated at the demand nodes. Hillsman and Rhoda (1978) identified three classes of aggregation errors: source A, B and C errors. Source A errors arise due to the approximation of the actual values of distance; source B errors are a particular case, that occurs when a demand point coincides with a candidate location and the distance between the demand and the potential facility is considered equal to zero; and source C errors correspond to an incorrect assignment of the demands to facilities.

Another contribution to the analysis of the *p*-median was that of Kariv and Hakimi (1979), who proved that the general *p*-median, where p is a variable, is NP-hard even in the case of a planar network of maximum vertex degree 3, with vertices of weight 1 and edges of length 1. Also, they rediscovered Hua-Lo Keng and others (1962) algorithm for locating one median on a network, and proposed an $O(n^2p^2)$ algorithm to find more than one median on trees.

There are several reviews of results related to the *p*-median. The first review focusing on the p-center and the p-median problems was that of Tansel et al. (1983a and 1983b). After a classification of the location problems on networks, the authors describe different variants of the problem, as well as solution techniques, and end with results specific o tree networks. Generalizations and extensions of the *p*-median are covered in the excellent review by Mirchandani (1990): the multicommodity *p*-median of Hakimi and Maheshwari (1972), in which there are different amounts of demand for different products or commodities, as well as different routes for each product; generalizations that consider some type of constraint (facility capacity, arc capacity, distance constraints and implementation constraints); generalizations considering probabilistic travel distances and demands; oriented and non-oriented networks; nonlinear transportation costs; and hierarchical *p*-median. Later, Marianov and Serra (2002) review some of the applications of the *p*-median model, focusing on those in the public sector. Other two reviews are oriented specifically to solution methods: Mladenović et al.

(2007) survey metaheuristic approaches to the solution of the p-median, while Reese (2006) presents an annotated bibliography of solution methods.

Finally, the reviews of Brandeau and Chiu (1989), Hale and Moberg (2003) and Snyder (2006), include material about the *p*-median, although the goal of the two first ones is to overview the research on location problems, while Snyder synthesizes the work available on facility location under uncertainty.

Potential future directions

Hakimi, as nearly everybody else, assumes the each demand travels, or is connected to its allocated facility, through the shortest possible path, which is the only logical choice in the case of a communications switch. However, in general, it is likely that more than one path exists between two points, and in some cases, the chosen path could not necessarily be the shortest. Furthermore, customers can choose different paths at different times, and this is a situation that is worth investigating.

When multiple facilities are being located, the allocation could be done following different criteria, including distance, but not excluding equity in facility workload, the degree in what each facility fits some characteristics of the demand node, and many more. In the traditional uncapacitated p-median problem of Hakimi (1965), each demand is allocated to its closest facility. This allocation policy has different meanings, depending on the "freedom" of the customers (demand) to choose the facility to patronize. A first case corresponds to a "dictatorial" system, in which there is a planner that allocates and enforces the observance of the allocation. An example of this case is the location of telephone switches: the telephone company locates the switches and connects subscribers to the closest switch. Note that if the planner cannot enforce in practice the observance of the allocation, the solution of the problem will not necessarily be a valid model of the real system, and it will then be suboptimal.

The second case corresponds to that in which customers, being free to choose the facility to patronize, select always the closest facility. Indeed, this is not necessarily always true, as Drezner and Drezner (2007) noted.

We remark also that, in this theorem, as well as in the definition of absolute median, it is assumed that demand is concentrated at the nodes of the graph, rather than being distributed on the entire graph (nodes and edges).

Conclusions

Since the early works of Hakimi and ReVelle, The P-Median problem has been, and still is, one of the most studied models in the facility location allocation academic literature, not only from a theoretical point of view, but also form its complexity in finding solution methods. Nevertheless, despite its potentiality for solving applied location problems and helping as a decision support tool to make decisions on locations, only a few real world applications can be found published in the academic literature, compared to the existing amount of theoretical papers.

The p-median problem and its extensions are useful to model many real world situations, such as the location of industrial plants, warehouses and public facilities (see, for example, Christofides, 1975, for a list of applications). The P-median problem can also be interpreted in terms of cluster analysis; locations of users are then replaced by points in an m-dimensional space (see Hansen and Jaumard, 1997, for a survey of cluster analysis from a mathematical programming viewpoint). It may thus offer a powerful tool for data mining applications (Ng and Han, 1994). Other applications of the p-median problem are related to the formation of cells (Ablasi et al. 2007), to the detection of glaucoma tests (Kolesar 1980), to the optimal sampling of biodiversity (Hortel and Lobo 2005), and to the assortment and trim loss minimization in the glass industry (Arbib and Marinelli 2004), among others.

References

Balinski, M.L., 1965, "Integer Programming: Methods, Uses and Computation", *Management Science* 12, 253-313.

Berman O and Larson R C, 1982, The Median Problem with Congestion, *Computers and Operations Research* 4(2) 119-126

Brandeau M L and Chiu S S, 1989, An Overview of Representative Problems in Location Research, *Management Science*, 35(6), 645-674

Calvo A and Marks H, 1973, Location of Health Care Facilities: An Analytical Approach, *Socio-economic Planning Sciences* 7, 407-422.

Cooper, L. (1963). Location–Allocation Problems, *Operations Research* 11, 331–343

Cooper, L. (1964). Heuristic Methods for Location–Allocation Problems, *SIAM Review* 6, 37–53

Drezner Z, 1987, Heuristic Solution Methods for Two Location Problems with Unreliable Facilities, *The Journal of the Operational Research Society*, 38(6), 509-514

Drezner T and Drezner Z, 2007, The gravity p-median model, *European Journal of Operational Research*, 179, 1239–1251

Efroymson M A and Ray T L, 1966, A Branch-Bound Algorithm for Plant Location, *Operations Research*, 14(3), 361-368

Frank H, 1966, Optimum locations on a graph with probabilistic demands, *Operations Research* 14(3), 409-421.

Goldman A J, 1969, Optimal locations for centers in a network, *Transportation Science* 3(2), 352-360.

Goldman A J, 1971, Optimal center location in simple networks, *Transportation Science* 5(2), 212-221.

Gülicher, H. 1965 "Einige Eigenschaften optimaler Standorte in Verkehrsnetzen," *Schriften des Vereins für Socialpolitik* 42, 111-137.

Hakimi S L, 1964, Optimal location of switching centers and the absolute centers and medians of a graph, *Operations Research* 12, 450– 459.

Hakimi S L, 1965. Optimum distribution of switching centers in a communication network and some related graph theoretic problems. *Operations Research* 13, 462–475.

Hakimi S L and Maheshwari S N, 1972, Optimum Locations of Centers in Networks, *Operations Research*, 20(5), 967-973.

Hale T S and Moberg C R, 2003, Location Science Research: A Review, *Annals of Operations Research* 123, 21–35

Hillsman E and Rhoda R,1978, Errors in measuring distance from populations to service centers, *Annals of the Regional Science Association*, 12, 74-88.

Holmes J, Williams F and Brown L, 1972, Facility location under maximum travel restriction: An example using day care facilities, *Geographical Analysis*, 4, 258-266.

Hortel J. and J.M. Lobo, 2005, An ED. Based protocol of optimal sampling of biodiversity. Biodiversity and Conservation 14(12), 2913-2947.

Hua Lo-Keng and others, 1962, Application of Mathematical Methods to Wheat Harvesting, Chinese Mathematics, 2, 77-91.

Kariv O and Hakimi S L, 1979, An Algorithmic Approach to Network Location Problems. II: The P medians, *SIAM Journal on Applied Mathematics*, 37, 539–560.

Kuehn A A and Hamburger M J, 1963, A heuristic program for locating warehouses, *Management Science* 9 (4), 643–666.

Land A H and Doig A G, 1960, An automatic method for solving discrete programming problems, *Econometrica*, 28, 497-520

Maranzana F, 1964, On the Location of Supply Points to Minimize Transport Costs, *Operations Research Quarterly*, 15, 261-270

Marianov V and Serra D, 2002, Location Problems in the Public Sector, in *Facility Location: Applications and Theory*, Drezner Z and Hamacher H (eds), Springer, Berlin.

Minieka E, 1977, The Centers and Medians of a Graph, *Operations Research*, 25, 641–650.

Mirchandani P B, 1990, The p-median Problem and Generalizations, in *Discrete Location Theory*, Mirchandani, P.B., Francis, R.L. (Eds.), Wiley-Interscience, NY.

Mirchandani P B and Odoni A R, 1979, Locations of medians on stochastic networks, *Transportation Science*, 13(2) 85-97.

Mladenović N, Brimberg J, Hansen P, Moreno-Pérez J A, 2007, The p-median problem: A survey of metaheuristic approaches, *European Journal of Operational Research*, 179, 927–939

Morris, J., 1978, "On the extent to which certain fixed-charge depot location problems can be solved by LP." *Journal of the Operational Research Society*, 29, 71-76.

Narula S C, 1984, Hierarchical Location – Allocation Problems: a Classification Scheme, *European Journal of Operational Research*, 15, 183-189.

Reese J, 2006, Solution Methods for the p-Median Problem: An Annotated Bibliography, *Networks*, 48(3), 125–142

ReVelle C S, Swain R W, 1970, Central Facilities Location, *Geographical Analysis*, 2, 30-42.

Rosing, K., ReVelle C., and H. Rosing-Vogelaar, 1979, "The p-median model and its linear programming relaxation: an approach to large problems", *Journal of the Operational Research Society* 30(9), 815-823.

Serra D and Marianov V, 1998, The p-median problem in a changing network: The case of Barcelona, *Location Science*, 6, 383-394.

Snyder L V, 2006, Facility Location under Uncertainty: A Review, *IIE Transactions* 38(7), 537-554

Tansel B C, Francis R L and Lowe T J, 1983a Location on Networks: A Survey. Part I: The p-Center and p-Median Problems, *Management Science*, 29(4), 482-497.

Tansel B C, Francis R L and Lowe T J, 1983b Location on Networks: A Survey. Part II: Exploiting Tree Network Structure, *Management Science*, 29(4), 498-511.

Teitz M and Bart P, 1968, Heuristic Methods for Estimating the Generalized Vertex Median of a Weighted Graph, *Operations Research*, 16, 955-961.

Toregas C, Swain R W, ReVelle C S and Bergman L, 1971, The Location of Emergency Service Facilities, *Operations Research*, 19, 1363-1373.

Wesolowsky G O and Truscott W G, 1975, The Multiperiod Location-Allocation Problem with Relocation of Facilities, *Management Science*, 22(1), 57-65.

********************POSSIBLE INTERESTING PAPERS*******************

Church RL, Garfinkel RS. Locating an obnoxious facility on a network. Transportation Science 1978;12(2):107–18 (does it have to do with p-median?)

Competitive (I can't get the Hakimi paper in Annals of OR, 1986, "p-median theorems for competitive locations". I don't know if it has to do with p-median)

Optimal facility location with multi-purpose trip making. by Suzuki, Tsutomu and Hodgson, M. John IIE Transactions • May, 2005 •