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THE M-COMPONENTS OF LEVEL SETS OF CONTINUOUS FUNCTIONS IN WBV

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Abstract

We prove that the topographic map structure of upper semicontinuous functions, defined in terms of classical connected components of its level sets, and of functions of bounded variation (or a generalization, the WBV functions), defined in terms of M-connected components of its level sets, coincides when the function is a continuous function in WBV. Both function spaces are frequently used as models for images. Thus, if the domain $\overline{\Omega}$ of the image is Jordan domain, a rectangle, for instance, and the image $u \in C(\overline{\Omega}) \cap WBV(\Omega)$ (being constant near $\partial\Omega$), we prove that for almost all levels λ of u, the classical connected components of positive measure of $[u \geq \lambda]$ coincide with the M-components of $[u \geq \lambda]$. Thus the notion of M-component can be seen as a relaxation of the classical notion of connected component when going from $C(\overline{\Omega})$ to $WBV(\Omega)$.

1. Introduction

An image can be realistically modelled as a real function $u(x)$ where x represents an arbitrary point of $\mathbb{R}^N$ ($N = 2$ for usual snapshots, 3 for medical images or movies) and $u(x)$ denotes the grey level at x. We shall assume that the image domain is finite (a hyperrectangle). Tipically $u(x)$ represents the photonic flux over a wide band of wavelengths and we have a proper grey level image. Now, $u(x)$ may also represent a colour intensity, when the photonic flux is subjected to a colour selective filter. In the following, we always consider scalar images, that is, images with a single channel, be it colour or grey level.

According to Mathematical Morphology, an image u is a representative of an equivalence class of images v obtained from u via a contrast change, i.e., $v = g(u)$ where g, for simplicity, will be a continuous strictly

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increasing function [**40**], [**19**]. The contrast of an image depends on the sensor's properties, on the lightning conditions, on the objects' reflection properties, etc., and these conditions are unknown. This led the physicist and gestaltist M. Wertheimer [**47**] to state as a principle that the grey level is not an observable. Images are observed up to an arbitrary and unknown contrast change. Mathematical Morphology recognized contrast invariance as a basic invariance requirement and proposed that image analysis operations should take into account this invariance principle, [**40**]. Under this assumption, an image is characterized by its upper (or lower) level sets $X_\lambda = \{x : u(x) \geq \lambda\}$ (resp. $X'_\lambda = \{x : u(x) \leq \lambda\}$). Moreover, the image can be recovered from its level sets by the reconstruction formula

$$u(x) = \sup\{\lambda : x \in X_\lambda\}.$$

As it is easily seen, the family of level sets (upper or lower) of u is invariant under continuous strictly increasing contrast changes. An image operator T is contrast invariant if

$$T(g(u)) = g(T(u)),$$

for any continuous strictly increasing contrast change g and any image u. In particular, many efficient denoising operators respect this principle. For a classification of contrast invariant image multiscale smoothing operators we refer to [**1**], [**19**], [**35**].

Level sets are therefore basic objects for image processing and analysis. In order to have a more local description of the basic objects of an image, several authors [**40**], [**7**] proposed to consider the connected components of (upper or lower) level sets as the basic objects of the image. They argue that contrast changes are local and depend upon the reflectance properties of objects. Thus, not only global contrast, but also local contrast is irrelevant. In [**7**], a notion of local contrast change is defined and it is proved that only connected components of level sets are invariant under such contrast changes. This led to the introduction of topographic maps, the family of connected components of upper (or lower) level sets $[u \geq \lambda]$ (resp. $[u \leq \lambda]$). More generally, we can consider the family of sets generated by connected components of sections of u, i.e., sets of the form $[\lambda \leq u \leq \mu]$, and their countable unions and intersections. This was at the basis of [**6**] where the authors compare different satellite images of the same landscape, taken at different times (or in different channels) and at the basis of the approach in [**29**] to image registration, one of the most basic tools in multiimage processing. Image registration based on connected components of level sets is shown to work efficiently where classical correlation techniques fail: when both registered images do not correspond to almost simultaneous snapshots.

The mathematical description of the topographic map requires a functional model for images. Mainly, two models for images are frequently used: the image modelled as an upper semicontinuous function [**40**], [**19**], or as a function of bounded variation, the so-called, BV-model [**36**], [**2**]. The Mathematical Morphology school models functions as upper semicontinuous functions and the filters map such functions into such functions [**40**], [**39**]. This model has also recently been used to develop an algorithm to merge both the connected components of upper and lower level sets [**30**], [**31**]. Indeed, to be able to merge these trees into a single tree, P. Monasse [**30**] defines a shape by filling-in the holes of a connected component of an upper or lower level set. The tree of upper shapes can be merged with the tree of lower shapes into a single tree. This tree gives acces in a fast way to the shapes of the image and appears to be useful for many applications: a fast implemementation of grain filters [**30**], [**31**], and a registration algorithm [**30**], [**29**], to name a few of them. In a different direction, we carried [**4**] an analysis of the topographic map to compute the changes of topology of the system of level lines of an upper semicontinuous function, and we proved that after applying L. Vincent filters the changes of topology happen at a finite number of levels. The union of both previous analysis gives a sort of Morse theory for this class of functions [**5**]. The other basic model is the so-called BV-model, introduced for the purpose of image denoising by L. I. Rudin, S. Osher and E. Fatemi in [**36**], in which images are functions of bounded variation. The BV-model is a sound model for images which have discontinuities and has become a popular model for image restoration [**36**], [**9**], [**12**], [**46**] and edge detection [**10**]. Since, the class of BV functions is not invariant under contrast changes, in [**2**], the authors introduced the WBV-model in which images are functions whose level sets are sets of finite perimeter (modulo a null set of levels). This class of functions contains the functions of bounded variation and is invariant under contrast changes. Indeed, it coincides with the set of functions u such that $g(u)$ is a function of bounded variation for a suitable bounded continuous strictly increasing contrast change g. In this model, and with a suitable notion of connectedness, called M-conectedness in [**2**], [**26**], the M-connected components of upper (or lower) level sets are rectifiable and can be described in terms of their boundaries by means of a generalized Jordan curve theorem. Indeed, in $\mathbb{R}^2$, the boundary of an M-connected set is described by a family, countable, at most, of Jordan curves, and it can be reconstructed from them. This mathematical model is also a sound foundation for a geometric description of the shapes of the image and as a function space to define connected filters [**42**], [**39**], [**2**], [**26**], i.e., filters that simplify the topographic map of the image.

Our purpose in this paper is to prove that both models and descriptions are the same when the image is a continuous one in the WBV model. Indeed, if the domain of the image is Jordan domain (see the text for definitions), a rectangle, for instance, and the image $u \in C(\overline{\Omega}) \cap WBV(\Omega)$ (being constant near $\partial\Omega$), we prove that for almost all levels λ of u, the classical connected components (of positive measure) of $[u \geq \lambda]$ coincide with the M-components of $[u \geq \lambda]$. Thus the notion of M-component [**2**] can be seen as a relaxation of the classical notion of connected component when going from $C(\overline{\Omega})$ to $WBV(\Omega)$. To prove the above result, our strategy will be: we take λ such that $[u \geq \lambda]$ is a set of finite perimeter in $\mathbb{R}^N$, if a classical connected component X of $[u \geq \lambda]$ is not M-connected, then the same thing happens for a simplification of u with the help of L. Vincent filters, i.e., for $w_\epsilon = IS_\epsilon SI^\epsilon u$, $\epsilon \in \{1/n : n \geq 1\}$ (see the text for definitions). Then we observe that the topology of the level sets of w_ϵ changes at λ. We prove that this may happen only a finite number of times for w_ϵ. Since $\epsilon \in \{1/n : n \geq 1\}$, this discards countably many levels. These levels together with the levels for which $[u \geq \lambda]$, or $[IS_\epsilon SI^\epsilon u \geq \lambda]$, is not a set of finite perimeter is a null set of levels. Thus, for the other levels λ, the classical connected components of $[u \geq \lambda]$ (of positive measure) are also M-connected sets. Moreover, we prove (under some conditions that hold for almost all level set of u) that a suitable representative of an M-connected set is connected. This is essentially the proof contained in this paper. For future reference, we shall also develop in some detail the properties of L. Vincent filters IS_ϵ, SI^ϵ (see Section 3).

After completing this work, we became aware of the work [**21**] on functions of two variables. In this paper the author gives a Morse theory for continuous functions of two variables and, in particular, he describes the singularities of level sets of continuous functions of two variables. What he calls level sets are sets of the form $\{x \in D : F(x) = t\}$, where $t \in \mathbb{R}$ and F is a continuous function defined in a domain D of the plane. For $N = 2$, the result described in last paragraph could be deduced from the results in Kronrod's paper [**21**].

This paper is organized as follows. Section 2 introduces the notion of maximal monotone section, the largest sections of the topographic map where no change of topology occurs for the level sets of the upper topographic map. Section 3 proves some basic facts about some filters of L. Vincent type on continuous functions. Section 4 is devoted to prove that these L. Vincent filters produce an effective simplification of the topographic map, indeed the filtered continuous images have a finite number of maximal monotone sections, a mathematical translation of the idea that there is a finite number of topological changes in the upper topographic map. We start Section 5 by recalling the basic facts and results proved in [**2**] about connected components of sets of finite perimeter and the WBV model. Then we prove that if $u \in C(\overline{\Omega}) \cap WBV(\Omega)$ (being constant near $\partial\Omega$), for almost all levels λ of u, the classical connected components (of positive measure) of $[u \geq \lambda]$ coincide with the M-components of $[u \geq \lambda]$.

2. Monotone sections of the topographic map

Let D be a subset of $\mathbb{R}^N$. Given a function $u\colon D \to \mathbb{R}$, we call upper (lower) level set of u any set of the form $[u \geq \lambda] := \{x \in D : u(x) \geq \lambda\}$ or $[u > \lambda] := \{x \in D : u(x) > \lambda\}$ ($[u \leq \lambda] := \{x \in D : u(x) \leq \lambda\}$ or $[u < \lambda] := \{x \in D : u(x) < \lambda\}$) where $\lambda \in \mathbb{R}$. The (upper) topographic map of a function u is the family of the connected components of the level sets of u, $[u \geq \lambda]$, $\lambda \in \mathbb{R}$, the connected components being understood in the relative topology of D. It was proved in [**7**] that the topographic map is the structure of the image which is invariant under local contrast changes, a notion also defined in [**7**]. We shall study the structure of the topographic map for continuous functions (a similar study can be done for bounded upper semicontinuous functions). We shall define a notion of nonsingular region of the topographic map trying to express the fact that the level lines of the topographic map in a nonsingular region are homotopic, as it happens for smooth functions where singularities are understood in the usual way [**28**]. A first version of this notion appeared in [**8**].

Let $u\colon D \to \mathbb{R}$ be a function. For each $\lambda, \mu \in \mathbb{R}$, $\lambda \leq \mu$ we define

$$U_{\lambda,\mu} = \{x \in D : \lambda \leq u(x) \leq \mu\}.$$

Definition 1. Let $u\colon D \to \mathbb{R}$ be a continuous function. A monotone section of the topographic map of u is a set of the form

$$X_{\lambda,\mu} = \mathrm{cc}(U_{\lambda,\mu}), \tag{1}$$

for some $\lambda, \mu \in \mathbb{R}$ with $\lambda \leq \mu$, such that for any $\lambda', \mu' \in [\lambda, \mu]$, $\lambda' \leq \mu'$ the set

$$\{x \in X_{\lambda,\mu} : \lambda' \leq u(x) \leq \mu'\}$$

is a connected component of $U_{\lambda',\mu'}$.

Our next result permits us to define a monotone section which is maximal with respect to inclusion. Those sets are the non singular sets we mentioned above.

Proposition 1. *Assume that $u\colon D \to \mathbb{R}$ is a continuous function such that for each $\lambda, \mu \in \mathbb{R}$ with $\lambda \leq \mu$ the set $U_{\lambda,\mu}$ has a finite number of connected components. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. Then, if X_{λ_1,λ_2}, X_{μ_1,μ_2} are monotone sections such that $X_{\lambda_1,\lambda_2} \cap X_{\mu_1,\mu_2} \neq \emptyset$, then $X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}$ is also a monotone section. In other words, the union of intersecting monotone sections is a monotone section.*

Proof: We observe that $[\lambda_1, \lambda_2] \cap [\mu_1, \mu_2] \neq \emptyset$ since $X_{\lambda_1,\lambda_2} \cap X_{\mu_1,\mu_2} \neq \emptyset$. Let $\beta_1 = \min(\lambda_1, \mu_1)$, $\beta_2 = \max(\lambda_2, \mu_2)$. It is easy to see that $X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}$ is a connected component of U_{β_1,β_2}. Indeed, if $\beta_1 = \lambda_1$ and $\beta_2 = \lambda_2$ then $X_{\mu_1,\mu_2} \subseteq X_{\lambda_1,\lambda_2}$ and the conclusion follows. Without loss of generality, we may assume that $\beta_1 = \lambda_1$ and $\beta_2 = \mu_2$, i.e., $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2$. Let V_{λ_1,μ_2} be the connected component of U_{λ_1,μ_2} containing $X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}$. Observe that X_{λ_1,λ_2}, X_{μ_1,μ_2} are closed sets in D. Let Z_{λ_1,λ_2}, Z_{μ_1,μ_2} be finite unions of connected components of U_{λ_1,λ_2}, U_{μ_1,μ_2}, respectively, such that $U_{\lambda_1,\lambda_2} = X_{\lambda_1,\lambda_2} \cup Z_{\lambda_1,\lambda_2}$, $U_{\mu_1,\mu_2} = X_{\mu_1,\mu_2} \cup Z_{\mu_1,\mu_2}$, $X_{\lambda_1,\lambda_2} \cap Z_{\lambda_1,\lambda_2} = \emptyset$, $X_{\mu_1,\mu_2} \cap Z_{\mu_1,\mu_2} = \emptyset$. Since Z_{λ_1,λ_2} is a closed set in U_{λ_1,λ_2} and this set is closed in D, Z_{λ_1,λ_2} is also a closed set in D. Similarly, Z_{μ_1,μ_2} is a closed set in D. Then the sets $Y_{\lambda_1,\lambda_2} = V_{\lambda_1,\mu_2} \cap Z_{\lambda_1,\lambda_2}$ and $Y_{\mu_1,\mu_2} = V_{\lambda_1,\mu_2} \cap Z_{\mu_1,\mu_2}$ are closed sets in V_{λ_1,μ_2}. Now, observe that $(X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}) \cap (Y_{\lambda_1,\lambda_2} \cup Y_{\mu_1,\mu_2}) = \emptyset$. Let us check that $Y_{\mu_1,\mu_2} \cap X_{\lambda_1,\lambda_2} = \emptyset$. Indeed $X_{\lambda_1,\lambda_2} \cap X_{\mu_1,\mu_2} \subseteq \{z \in X_{\lambda_1,\lambda_2} : \mu_1 \leq u(x) \leq \mu_2\} = \{z \in X_{\lambda_1,\lambda_2} : \mu_1 \leq u(x) \leq \lambda_2\}$ the last set being a connected component Q of U_{μ_1,λ_2}. In particular, we observe that $X_{\mu_1,\mu_2} \cap Q \neq \emptyset$ and we conclude that $Q \subseteq X_{\mu_1,\mu_2}$. Now, observe that $Y_{\mu_1,\mu_2} \cap X_{\lambda_1,\lambda_2} \subseteq Q$. Indeed, if $p \in Y_{\mu_1,\mu_2} \cap X_{\lambda_1,\lambda_2}$, then $p \in \{z \in X_{\lambda_1,\lambda_2} : \mu_1 \leq u(x) \leq \mu_2\} = Q$. Since $Q \subseteq X_{\mu_1,\mu_2}$, and $Y_{\mu_1,\mu_2} \subseteq Z_{\mu_1,\mu_2}$, this implies that $Y_{\mu_1,\mu_2} \cap X_{\lambda_1,\lambda_2} = \emptyset$. Similarly, $Y_{\lambda_1,\lambda_2} \cap X_{\mu_1,\mu_2} = \emptyset$. Since

$$V_{\lambda_1,\mu_2} = (X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}) \cup (Y_{\lambda_1,\lambda_2} \cup Y_{\mu_1,\mu_2})$$

and V_{λ_1,μ_2} is connected then $Y_{\lambda_1,\lambda_2} = \emptyset$, $Y_{\mu_1,\mu_2} = \emptyset$. It follows that

$$V_{\lambda_1,\mu_2} = X_{\lambda_1,\lambda_2} \cup X_{\mu_1,\mu_2}.$$

The proof that for any $\lambda, \mu \in [\beta_1, \beta_2]$, $\lambda \leq \mu$ the set

$$\{x \in V_{\beta_1,\beta_2} : \lambda \leq u(x) \leq \mu\}$$

is a connected component of $U_{\lambda,\mu}$ follows along the same lines of argument as the previous one. □

Let $x \in D$ and $\lambda = u(x)$. For each $\eta \geq \lambda$, let $X_{\lambda,\eta} = \mathrm{cc}(U_{\lambda,\eta}, x)$. We define

$$\eta_+(x,\lambda) = \sup\{\eta : \eta \geq \lambda \quad \text{s. t. } X_{\lambda,\eta} \text{ is a monotone section}\}.$$

Similarly, we define

$$\eta_-(x,\lambda) = \inf\{\eta : \eta \leq \lambda \quad \text{s. t. } X_{\eta,\lambda} \text{ is a monotone section}\}.$$

Note that both numbers are well defined since $X_{\lambda,\lambda}$ is always a monotone section. Note that, by definition, $\eta_-(x,\lambda) \leq \eta_+(x,\lambda)$. By Proposition 1, we may define the (open, closed, half-open, half-closed) interval $I(x,\lambda)$ containing λ whose end-points are $\eta_-(x,\lambda)$, $\eta_+(x,\lambda)$ and which determines a *monotone section containing x maximal with respect to inclusion*, which we denote by $X_{I(x,\lambda)}$. Note that $\lambda \in I(x,\lambda)$ for all $\lambda \in (-\infty, \sup_D u(x)]$. Both functions, $\eta_+(x,\lambda)$, $\eta_-(x,\lambda)$, are non-decresing functions of λ and have some precise behavior. We shall not give here a detailed description of them. Our purpose will be to prove that, if D is measurable and there exist some $\delta > 0$ such that for any $\lambda \in [a,b]$, $a = \inf u$, $b = \sup u$, and any $X = \mathrm{cc}(U_{a,\lambda})$ or $X = \mathrm{cc}(U_{\lambda,b})$ we have that $\mathrm{meas}(X) \geq \delta$, then there is only a finite number of maximal monotone sections in the topographic map of u.

3. Vincent-Serra filters

Let D be a measurable subset of $\mathbb{R}^N$. Let $u\colon D \to \mathbb{R}$ be a measurable function. Let us define two morphological operators, the Vincent-Serra operators, simplifying the topographic map of u by eliminating the small connected components of its upper and lower level sets. Let $\epsilon > 0$. Then, for each $x \in D$, we define

$$\mathcal{F}^\epsilon(u,x) = \{X = \mathrm{cc}([u \geq \lambda], x) : |X| \geq \epsilon\},$$
$$\mathcal{F}_\epsilon(u,x) = \{X = \mathrm{cc}([u < \lambda], x) : |X| > \epsilon\}.$$

Let us define the following Vincent-Serra operators

$$SI^\epsilon u(x) = \sup_{B \in \mathcal{F}^\epsilon(u,x)} \inf_{y \in B} u(y),$$
$$IS_\epsilon u(x) = \inf_{B \in \mathcal{F}_\epsilon(u,x)} \sup_{y \in B} u(y),$$

where we understand that $\sup_{B \in \mathcal{F}^\epsilon(u,x)} \inf_{y \in B} u(y) = -\infty$ if $\mathcal{F}^\epsilon(u,x) = \emptyset$ and $\inf_{B \in \mathcal{F}_\epsilon(u,x)} \sup_{y \in B} u(y) = +\infty$ if $\mathcal{F}_\epsilon(u,x) = \emptyset$.

Both operators can be described in terms of basis of structuring elements independent of u. Indeed, let $\mathcal{B}^\epsilon = \{B : B \text{ is connected}, 0 \in B, |B| \geq \epsilon\}$, $\mathcal{B}_\epsilon = \{B : B \text{ is connected}, 0 \in B, |B| > \epsilon\}$. Then

$$SI^\epsilon u(x) = \sup_{B \in x + \mathcal{B}^\epsilon} \inf_{y \in B} u(y), \tag{2}$$

$$IS_\epsilon u(x) = \inf_{B \in x + \mathcal{B}_\epsilon} \sup_{y \in B} u(y). \tag{3}$$

Let us denote, for the time being, the right hand side of (2) and (3) by $SI^{\epsilon,*}u(x)$, respectively, $IS_{\epsilon,*}u(x)$. Obviously, from the definition we have that $SI^\epsilon u(x) \leq SI^{\epsilon,*}u(x)$ and $IS_\epsilon u(x) \geq IS_{\epsilon,*}u(x)$. Now, if $SI^{\epsilon,*}u(x) < \infty$, given $\delta > 0$, let $B \in x + \mathcal{B}^\epsilon$ be such that $i_B := \inf_{y \in B} u(y) \geq SI^{\epsilon,*}u(x) - \delta$. Then $B \subseteq [u \geq i_B]$ and, therefore, $B \subseteq X := \mathrm{cc}([u \geq i_B], x)$. Hence $X \in \mathcal{F}^\epsilon(u,x)$ and

$$SI^{\epsilon,*}u(x) - \delta \leq i_B \leq \inf_{y \in X} u(y) \leq SI^\epsilon u(x).$$

In a similar way, we prove that $SI^\epsilon u(x) = \infty$ if $SI^{\epsilon,*}u(x) = \infty$. We have checked formula (2). In a similar way we prove the identity (3).

Proposition 2. *Assume that $u, v \colon D \to \mathbb{R}$ are measurable functions. Then*

i) *$IS_\epsilon u \geq u$ and $SI^\epsilon u \leq u$.*
ii) *If $u \leq v$, then $IS_\epsilon u \leq IS_\epsilon v$ and $SI^\epsilon u \leq SI^\epsilon v$.*
iii) *$IS_\epsilon(\alpha) = SI^\epsilon(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$.*
iv) *$IS_\epsilon(u+\alpha) = IS_\epsilon u + \alpha$ for all $\alpha \in \mathbb{R}$. A similar statement holds for SI^ϵ.*

Proof: i) Let $x \in D$. If $\mathcal{F}_\epsilon(u,x) = \emptyset$, then $IS_\epsilon u(x) = \infty$ and we are done. If $B \in \mathcal{F}_\epsilon(u,x)$, then $x \in B$, and $\sup_{y \in B} u(y) \geq u(x)$. Thus $IS_\epsilon u(x) \geq u(x)$. In the same way we prove that $SI^\epsilon u(x) \leq u(x)$.

The proof of ii) follows immediately from the identities (2) and (3). The proof of assertions iii) and iv) is immediate and we shall omit it. □

Proposition 3. *If $X = \mathrm{cc}([IS_\epsilon u < \lambda], x) \neq \emptyset$, then $|X| > \epsilon$. If $X = \mathrm{cc}([SI^\epsilon u \geq \lambda], x) \neq \emptyset$, then $|X| \geq \epsilon$. If $|\mathrm{cc}([u \geq \lambda], x)| \geq \epsilon$ (resp. $|\mathrm{cc}([u < \lambda], x)| \geq \epsilon$) then $|\mathrm{cc}([IS_\epsilon u \geq \lambda], x)| \geq \epsilon$ (resp. $|\mathrm{cc}([SI^\epsilon u < \lambda], x)| \geq \epsilon$).*

Remark 1. Thus, by filtering the small connected components of the upper and lower level sets with the Vincent-Serra operators we can guarantee that the assumption of Proposition 1 is satisfied (see Section 4).

Proof: Let $X = \mathrm{cc}([IS_\epsilon u < \lambda], x)$ $(\neq \emptyset)$. Since $x \in X$, by definition of $IS_\epsilon u$ there is a set $B_1 \in \mathcal{F}_\epsilon(u, x)$ such that

$$\sup_{y \in B_1} u(y) < \lambda.$$

Let $z \in B_1$. Observe that $B_1 \in \mathcal{F}_\epsilon(u, z)$. Then

$$IS_\epsilon u(z) = \inf_{B \in \mathcal{F}_\epsilon(u,z)} \sup_{y \in B} u(y) \leq \sup_{y \in B_1} u(y) < \lambda.$$

Hence $B_1 \subseteq [IS_\epsilon u < \lambda]$ and, as a consequence, $B_1 \subseteq \mathrm{cc}([IS_\epsilon u < \lambda], x)$. It follows that

$$|\,\mathrm{cc}([IS_\epsilon u < \lambda], x)| \geq |B_1| > \epsilon.$$

Now, since $IS_\epsilon u \geq u$ we have that $\mathrm{cc}([u \geq \lambda], x) \subseteq \mathrm{cc}([IS_\epsilon u \geq \lambda], x)$. Thus, if $|\,\mathrm{cc}([u \geq \lambda], x)| \geq \epsilon$ then $|\,\mathrm{cc}([IS_\epsilon u \geq \lambda], x)| \geq \epsilon$. The corresponding statements for SI^ϵ are proved in a similar way. □

Definition 2. Let X be a measurable subset of $\mathbb{R}^N$. Define

$$T_\epsilon X = \cup\{Y : Y \in \mathrm{CC}(X),\ |Y| \geq \epsilon\}$$

and

$$T'_\epsilon X = \cup\{Y : Y \in \mathrm{CC}(X),\ |Y| > \epsilon\}.$$

The components of X are closed subsets (in the relative topology) of X, hence measurable. Hence $T_\epsilon X$, $T'_\epsilon X$ are both measurable.

The following proposition is an easy consequence of last definition.

Proposition 4. *Both operators T_ϵ and T'_ϵ are nondecreasing on measurable subsets of $\mathbb{R}^N$.*

Proposition 5. i) *T_ϵ is upper continuous on compact sets, i.e., if X_ρ is a non-increasing family of compact sets and $X = \cap_{\rho>0} X_\rho$, then $T_\epsilon X = \cap_{\rho>0} T_\epsilon X_\rho$.*

ii) *Suppose that D is a measurable subset of $\mathbb{R}^N$ which is locally compact and locally connected when endowed with the relative topology of $\mathbb{R}^N$. We suppose that the measure of the relative open sets of D is strictly positive. Then T'_ϵ is lower continuous on open sets, i.e., if O_ρ is a non-decreasing family of relative open subsets of D and $O = \cup_{\rho>0} O_\rho$, then*

$$T'_\epsilon O = \cup_{\rho>0} T'_\epsilon O_\rho. \tag{4}$$

Proof: i) From the monotonicity of T_ϵ it follows that $T_\epsilon X \subseteq \cap_{\rho>0} T_\epsilon X_\rho$. Now, let $p \in T_\epsilon X_\rho$ for all $\rho > 0$. Let $Y_\rho = \mathrm{cc}(X_\rho, p)$ be such that $|Y_\rho| \geq \epsilon$. Let $Y = \cap_{\rho>0} Y_\rho$. Then $Y = \mathrm{cc}(X, p)$. Indeed, $p \in Y \subseteq X$. Since Y

is an intersection of continua, i.e., compact connected sets, then Y is connected. Hence $Y \subseteq \mathrm{cc}(X,p)$. On the other hand, $\mathrm{cc}(X,p) \subseteq \mathrm{cc}(X_\rho,p)$ for all $\rho > 0$. Thus $\mathrm{cc}(X,p) \subseteq \cap_{\rho>0} Y_\rho = Y$ and we have the equality of both sets. In particular, $|Y| = \lim_{\rho\to 0} |Y_\rho| \geq \epsilon$ and we deduce that $p \in Y \subseteq T_\epsilon X$. We have proved that $T_\epsilon X = \cap_{\rho>0} T_\epsilon X_\rho$.

ii) Again, from the monotonicity of T'_ϵ we deduce that $\cup_{\rho>0} T'_\epsilon O_\rho \subseteq T'_\epsilon O$. Now, let $p \in T'_\epsilon O$. Then $Y = \mathrm{cc}(O,p)$ is such that $|Y| > \epsilon$. Let $Y_\rho = \mathrm{cc}(O_\rho, p)$. Now we prove that

$$Y = \cup_{\rho>0} Y_\rho. \tag{5}$$

Since $O_\rho \subseteq O$ for all $\rho > 0$, it follows that $\cup_{\rho>0} Y_\rho \subseteq Y$. Since D is locally connected, the components of open sets are open. In particular Y is open in D. Now, since D is locally compact and locally connected then Y is also locally compact and locally connected, and, obviously connected. Then any two points can be connected by a locally connected continuum ([**33**, Theorem. 9.3, p. 92]). In particular, given $q \in Y$, there is a continuum $K_{p,q} \subseteq Y$ containing p and q. Then, for $\rho > 0$ small enough, $K_{p,q} \subseteq O_\rho$. Then also $K_{p,q} \subseteq \mathrm{cc}(O_\rho,p) = Y_\rho$. Thus $q \in Y_\rho$. We have proved (5). Then $\lim_{\rho\to 0} |Y_\rho| = |Y| > \epsilon$. Hence $|Y_\rho| > \epsilon$ for $\rho > 0$ small enough. We conclude that $p \in Y_\rho \subseteq T'_\epsilon O_\rho$ and, therefore, we have proved the identity (4). □

Lemma 1. *Let $u\colon D \to \mathbb{R}$ be a measurable function, D being a measurable subset of $\mathbb{R}^N$. Then*

i) *For all $\lambda \in \mathbb{R}$ we have $[SI^\epsilon u > \lambda] \subseteq T_\epsilon[u \geq \lambda] \subseteq [SI^\epsilon u \geq \lambda]$. In particular, for all $\lambda \in \mathbb{R}$ (hence a.e.) such that $|[SI^\epsilon u = \lambda]| = 0$ we have that $T_\epsilon[u \geq \lambda] = [SI^\epsilon u \geq \lambda]$ a.e.*

ii) *For all $\lambda \in \mathbb{R}$ we have $[IS_\epsilon u < \lambda] \subseteq T'_\epsilon[u < \lambda] \subseteq [IS_\epsilon u \leq \lambda]$. In particular, for all $\lambda \in \mathbb{R}$ (hence a.e.) such that $|[IS_\epsilon u = \lambda]| = 0$ we have that $T'_\epsilon[u < \lambda] = [IS_\epsilon u < \lambda]$ a.e.*

Proof: i) Let $\lambda \in \mathbb{R}$ and $\mathcal{C}^+$, (resp. $\mathcal{C}^-$) denote the family of components X of $[u \geq \lambda]$ such that $|X| \geq \epsilon$ (resp. $|X| < \epsilon$). Since $SI^\epsilon u \leq u$ we have

$$\begin{aligned}[SI^\epsilon u > \lambda] &= [SI^\epsilon u > \lambda] \cap [u \geq \lambda] \\ &= \cup_{X\in\mathcal{C}^+} [SI^\epsilon u > \lambda] \cap X \cup \cup_{X\in\mathcal{C}^-} [SI^\epsilon u > \lambda] \cap X.\end{aligned}$$

Now, we observe that $[SI^\epsilon u > \lambda] \cap X = \emptyset$ for all $X \in \mathcal{C}^-$. Otherwise, there is some $p \in [SI^\epsilon u > \lambda] \cap X$. Hence, $\inf_{y\in B} u(y) \geq \lambda$ for some $B \in \mathcal{F}^\epsilon(u,p)$. Thus, $p \in B \subseteq [u \geq \lambda]$ and, therefore, $B \subseteq \mathrm{cc}([u \geq \lambda], p) = X$.

It follows that $|X| \geq |B| \geq \epsilon$, a contradiction. Our claim follows. Hence

$$[SI^\epsilon u > \lambda] = \cup_{X\in\mathcal{C}^+}[SI^\epsilon u > \lambda] \cap X \subseteq \cup_{X\in\mathcal{C}^+} X = T_\epsilon[u \geq \lambda].$$

To prove the other inclusion we observe that $X \in \mathcal{F}^\epsilon(u,p)$ for all $X \in \mathcal{C}^+$ and all $p \in X$. Hence, $SI^\epsilon u(p) \geq \lambda$ for all $p \in X$ and all $X \in \mathcal{C}^+$. We conclude that $T_\epsilon[u \geq \lambda] \subseteq [SI^\epsilon u \geq \lambda]$.

ii) Let $\lambda \in \mathbb{R}$ and $\mathcal{C}^+$, (resp. $\mathcal{C}^-$) denote the family of components X of $[u < \lambda]$ such that $|X| > \epsilon$ (resp. $|X| \leq \epsilon$). Since $IS_\epsilon u \geq u$ we have

$$\begin{aligned}[IS_\epsilon u < \lambda] &= [IS_\epsilon u < \lambda] \cap [u < \lambda] \\ &= \cup_{X\in\mathcal{C}^+}[IS_\epsilon u < \lambda] \cap X \cup \cup_{X\in\mathcal{C}^-}[IS_\epsilon u < \lambda] \cap X.\end{aligned}$$

Now, we observe that $[IS_\epsilon u < \lambda] \cap X = \emptyset$ for all $X \in \mathcal{C}^-$. Otherwise, there is some $p \in [IS_\epsilon u < \lambda] \cap X$. Hence, $\sup_{y\in B} u(y) < \lambda$ for some $B \in \mathcal{F}_\epsilon(u,p)$. Thus, $p \in B \subseteq [u < \lambda]$ and, therefore, $B \subseteq \mathrm{cc}([u < \lambda], p) = X$. It follows that $|X| \geq |B| > \epsilon$, a contradiction. Our claim follows. Hence

$$[IS_\epsilon u < \lambda] = \cup_{X\in\mathcal{C}^+}[IS_\epsilon u < \lambda] \cap X \subseteq \cup_{X\in\mathcal{C}^+} X = T'_\epsilon[u < \lambda].$$

To prove the other inclusion we observe that $X \in \mathcal{F}_\epsilon(u,p)$ for all $X \in \mathcal{C}^+$ and all $p \in X$. Hence, $IS_\epsilon u(p) \leq \lambda$ for all $p \in X$ and all $X \in \mathcal{C}^+$. We conclude that $T'_\epsilon[u < \lambda] \subseteq [IS_\epsilon u \leq \lambda]$. ☐

As a consequence of the above lemma we obtain

Proposition 6. *Let $u\colon D \to \mathbb{R}$ be a measurable function. Then*

$$SI^\epsilon u(x) = \sup\{\lambda \in \mathbb{R} : x \in T_\epsilon[u \geq \lambda]\},$$

and

$$IS_\epsilon u(x) = \inf\{\lambda \in \mathbb{R} : x \in T'_\epsilon[u < \lambda]\}.$$

Proposition 7. *Let $u\colon D \to \mathbb{R}$ be a measurable function.*

i) *Suppose that $[u \geq \lambda]$ are compact for all $\lambda > \inf u$. Then*

$$[SI^\epsilon u \geq \lambda] = T_\epsilon[u \geq \lambda] \tag{6}$$

for all $\lambda \in \mathbb{R}$.

ii) *Suppose that D is locally compact, locally connected and the measure of its open sets is > 0. Suppose that $[u < \lambda]$ is open in D for all $\lambda \in \mathbb{R}$, or in other words, u is upper semicontinuous. Then*

$$[IS_\epsilon u < \lambda] = T'_\epsilon[u < \lambda] \tag{7}$$

for all $\lambda \in \mathbb{R}$.

Proof: From Proposition 5 we know that $\{T_\epsilon[u \geq \lambda] : \lambda > \inf u\}$ is upper continuous and $\{T'_\epsilon[u < \lambda] : \lambda \in \mathbb{R}\}$ is lower continuous. From this it easily follows that (6) holds for all $\lambda > \inf u$ and (7) holds for all $\lambda \in \mathbb{R}$. But (6) also holds for all $\lambda \leq \inf u$ since in this case both sets are equal to D. □

Lemma 2. *Let $u: D \to \mathbb{R}$ be measurable.*

i) *Then* $\inf SI^\epsilon u = \inf u$.

ii) *We always have* $\inf u \leq \inf IS_\epsilon u$. *If we suppose that the sets $[u < \lambda]$ contain a neighborhood of infinity in $\mathbb{R}^N$, for any $\lambda > \inf u$, then* $\inf IS_\epsilon u = \inf u$.

Proof: i) From the definition of $SI^\epsilon u$ it is clear that $\inf SI^\epsilon u \geq \inf u$. On the other hand, since $SI^\epsilon u \leq u$, we have also $\inf SI^\epsilon u \leq \inf u$, hence the equality.

ii) Since $u \leq IS_\epsilon u$, we have that $\inf u \leq \inf IS_\epsilon u$. If $[u < \lambda]$ contains a neighborhood of infinity in $\mathbb{R}^N$ for any $\lambda > \inf u$, then it is easy to see from the definition of $IS_\epsilon u$ that $\inf IS_\epsilon u = \inf u$. □

Proposition 8. *Let $u: D \to \mathbb{R}$ be an upper semicontinuous function.*

i) *Suppose that D is locally compact, locally connected and the measure of its open sets is > 0. Then $IS_\epsilon u$ is upper semicontinuous.*

ii) *Suppose that D contains at most a finite number of connected components of area $\geq \epsilon$, and $[u \geq \lambda]$ is compact for all $\lambda > \inf u$. Then $SI^\epsilon u$ is upper semicontinuous. Moreover, the upper level sets $[SI^\epsilon u \geq \lambda]$ are compact in the following cases:*

a) *for all $\lambda \in \mathbb{R}$ if $[u \geq \lambda]$ are compact for all $\lambda \in \mathbb{R}$,*

b) *for all $\lambda > \inf u$. In this case, if $[u = \inf u]$ is a neighborhood of infinity, then $[SI^\epsilon u = \inf SI^\epsilon u]$ is also a neighborhood of infinity.*

Proof: i) According to Proposition 7, ii), $[IS_\epsilon u < \lambda] = T'_\epsilon[u < \lambda] = \cup_{X \in \mathrm{CC}([u<\lambda]),\, |X|>\epsilon} X$. Since the last union is a union of open sets, the set $[IS_\epsilon u < \lambda]$ is open for every $\lambda \in \mathbb{R}$. Hence $IS_\epsilon u$ is upper semicontinuous.

ii) According to Proposition 7, i), $[SI^\epsilon u \geq \lambda] = T_\epsilon[u \geq \lambda] = \cup_{X \in \mathrm{CC}([u\geq\lambda]),\, |X|\geq\epsilon} X$ for every $\lambda \in \mathbb{R}$. If $\lambda > \inf u$, the last union is a finite union of sets closed in D, and the set $[SI^\epsilon u \geq \lambda]$ is also closed in D. If $\lambda \leq \inf u$, then $[u \geq \lambda] = D$ and, by our assumption on D, $T_\epsilon D$ is closed in D. Hence $SI^\epsilon u$ is upper semicontinuous. The above formula also proves that the sets $[SI^\epsilon u \geq \lambda]$ are compact for any $\lambda \in \mathbb{R}$ such that $[u \geq \lambda]$ is compact. This proves assertion a). In particular, they are compact for any $\lambda > \inf u$. To prove the last assertion of b), observe that, from the inequality $SI^\epsilon u \leq u$ it follows that $[u = \inf u] \subseteq [SI^\epsilon u = \inf u]$. Hence, if $[u = \inf u]$ is a neighborhood of infinity, also is $[SI^\epsilon u = \inf SI^\epsilon u]$. □

Proposition 9. *Suppose that D is locally compact, locally connected and the measure of its open sets is > 0. Assume also that D contains at most a finite number of connected components of area $\geq \epsilon$. Let $u\colon D \to \mathbb{R}$ be an upper semicontinuous function such $[u \geq \lambda]$ is compact for all $\lambda > \inf u$. Then $IS_\epsilon SI^\epsilon u$ is upper semicontinuous.*

Remark 2. It will be clear from next proposition that, if D is compact or D is closed and contains a neighborhood of infinity, then the upper level sets of $IS_\epsilon u$ are compact for all $\lambda > \inf u$. Therefore, $SI^\epsilon IS_\epsilon u$ is also upper semicontinuous.

Proposition 10. *Suppose that D is closed, locally connected and the measure of its open sets is > 0. We also suppose that either D is compact or it contains a neighborhood of infinity. Let $u\colon D \to \mathbb{R}$ be a continuous function such that $[u \geq \lambda]$ is compact for all $\lambda > \inf u$. Then $IS_\epsilon u$ is continuous. The upper level sets $[IS_\epsilon u \geq \lambda]$ are compact for all $\lambda > \inf u$. In case D is not compact, we have $\inf IS_\epsilon u = \inf u$. Moreover, if $[u = \inf u]$ is either empty or contains a neighborhood of infinity, then $[IS_\epsilon u = \inf IS_\epsilon u]$ is also empty or it contains neighborhood of infinity, in consonance with the property for u.*

Remark 3. Being closed, the lower level sets $[IS_\epsilon u \leq \lambda]$ are compact for all $\lambda \in \mathbb{R}$ in case D is compact. Note, that, by assumption, the complement of an open bounded set in D is either compact or it contains a neighborhood of infinity.

Proof: We already know that $IS_\epsilon u$ is upper semicontinuous. Let us prove that the sets $[IS_\epsilon u \leq \lambda]$ are closed for all $\lambda \in \mathbb{R}$. Since $[IS_\epsilon u \leq \inf IS_\epsilon u] = \cap_{\rho>0}[IS_\epsilon u \leq \inf IS_\epsilon u + \rho]$, it is enough to prove that $[IS_\epsilon u \leq \lambda]$ are closed for all $\lambda > \inf IS_\epsilon u$. Thus, let $\lambda \in \mathbb{R}$. Now, we observe that

$$\begin{aligned}[IS_\epsilon u \leq \lambda] &= \cap_{\rho>0}[IS_\epsilon u < \lambda + \rho] \\ &= \cap_{\rho>0} T'_\epsilon [u < \lambda + \rho] \\ &= \cap_{\rho>0} \cup \{X : X \in \mathrm{CC}([u < \lambda + \rho]),\ |X| > \epsilon\}.\end{aligned}$$

We claim that

$$\begin{aligned}(8)\quad \cap_{\rho>0} \cup\{X : X \in \mathrm{CC}([u < \lambda + \rho]),\ |X| > \epsilon\} \\ = \cup\{Y : Y \in \mathrm{CC}([u \leq \lambda]),\ |Y| \geq \epsilon\}.\end{aligned}$$

Let $Y \in \mathrm{CC}([u \le \lambda])$ be such that $|Y| \ge \epsilon$. Let $Y_\rho = \mathrm{cc}([u < \lambda + \rho], Y)$, $\rho > 0$. Since u is continuous, Y is closed and, using that D is locally connected, Y_ρ is an open set, being the connected component of an open set. Since $Y \subseteq Y_\rho$ and using that the measure of the open sets of D is > 0, we have that $\epsilon \le |Y| < |Y_\rho|$. Thus $Y \subseteq \cap_\rho Y_\rho$ where $Y_\rho \in \{X : X \in \mathrm{CC}([u < \lambda + \rho]), |X| > \epsilon\}$. Hence

$$\cup \{Y : Y \in \mathrm{CC}([u \le \lambda]), |Y| \ge \epsilon\} \subseteq \cap_{\rho>0} \cup \{X : X \in \mathrm{CC}([u < \lambda + \rho]), |X| > \epsilon\}.$$

Now, let $p \in \cap_{\rho>0} \cup \{X : X \in \mathrm{CC}([u < \lambda + \rho]), |X| > \epsilon\}$. Then for each $\rho > 0$, there is $X(p, \rho) = \mathrm{cc}([u < \lambda + \rho], p)$ such that $|X(p, \rho)| > \epsilon$. Observe that $X(p, \rho)$ is nondecreasing with ρ. Let $Y_p = \cap_{\rho>0} \mathrm{cc}([u < \lambda + \rho], p)$. We have

$$\begin{aligned} \mathrm{cc}([u \le \lambda], p) &\subseteq \cap_{\rho>0} \mathrm{cc}([u < \lambda + \rho], p) \\ &\subseteq \cap_{\rho>0} \mathrm{cc}([u \le \lambda + \rho], p) \subseteq [u \le \lambda]. \end{aligned} \tag{9}$$

Let $\lambda > \inf u$. Since $[u \ge \eta]$ is compact for all $\eta > \inf u$, $[u > \eta]$ is open in D and bounded for all $\eta > \inf u$. Since the complement of an open bounded set is either compact, in case D is, or a neighborhood of infinity, in case D contains a neighborhood of infinity, we know that $[u \le \lambda + \rho]$ is either compact or it contains a neighborhood of infinity for any $\rho > 0$. Hence $\mathrm{cc}([u \le \lambda + \rho], p)$ is also compact or contains a neighborhood of infinity. In case $\mathrm{cc}([u \le \lambda + \rho], p)$ are compact, we have

$$\mathrm{cc}([u \le \lambda], p) = \cap_{\rho>0} \mathrm{cc}([u \le \lambda + \rho], p).$$

In case $\mathrm{cc}([u \le \lambda + \rho], p)$ contain a neighborhood of infinity (note that it is equivalent to state this for all $\rho > 0$ or for some sequence $\rho_n \downarrow 0$), since $[u \ge \lambda]$ is compact (here we used that $\lambda > \inf u$), we have that $[u \le \lambda]$ is a neighborhood of infinity. Let Q be the connected component of $[u \le \lambda]$ which is a neighborhood of infinity. Then Q is contained in $\mathrm{cc}([u \le \lambda + \rho], p)$ for all $\rho > 0$. Let $\overline{\mathbb{R}^N}$ be the one-point compactification of $\mathbb{R}^N$. Let $\mathrm{cc}_{\overline{\mathbb{R}^N}}([u \le \lambda + \rho], p)$ be the component of $[u \le \lambda + \rho]$ in $\overline{\mathbb{R}^N}$ which contains p. Since $[u \le \lambda + \rho]$ is closed in $\mathbb{R}^N$, then $\mathrm{cc}_{\overline{\mathbb{R}^N}}([u \le \lambda + \rho], p)$ is compact in $\overline{\mathbb{R}^N}$ and contains Q. Then

$$\cap_{\rho>0} \mathrm{cc}_{\overline{\mathbb{R}^N}}([u \le \lambda + \rho], p)$$

is connected, and also

$$\cap_{\rho>0} \mathrm{cc}([u \le \lambda + \rho], p) = \cap_{\rho>0} \mathrm{cc}_{\overline{\mathbb{R}^N}}([u \le \lambda + \rho], p) \setminus \{\infty\}$$

is connected. From (9), it follows that $\mathrm{cc}([u \leq \lambda], p) = \cap_{\rho>0} \mathrm{cc}([u \leq \lambda + \rho], p)$. Hence, from the previuos discussion we know that

$$|\mathrm{cc}([u \leq \lambda], p)| = \lim_{\rho \to 0} |\mathrm{cc}([u \leq \lambda + \rho], p)| \geq \epsilon.$$

Thus, $p \in \cup\{Y : Y \in \mathrm{CC}([u \leq \lambda]), |Y| \geq \epsilon\}$. We have proved that

$$\cap_{\rho>0} \cup\{X : X \in \mathrm{CC}([u < \lambda + \rho]), |X| > \epsilon\} \subseteq \cup\{Y : Y \in \mathrm{CC}([u \leq \lambda]), |Y| \geq \epsilon\}.$$

Together with the opposite inclusion, this gives (8). Observe that, under any of our assumptions on D, the right hand side of (8) is a finite union of closed sets in $\mathbb{R}^N$, hence $[IS_\epsilon u \leq \lambda]$ is also a closed set in $\mathbb{R}^N$. In particular, we deduce that $IS_\epsilon u$ is continuous.

If D is compact, then all level sets of $IS_\epsilon u$, lower or upper, are compact. If D is not compact, then we deduce that all sets $[u < \lambda]$ contain a neighborhood of infinity for all $\lambda > \inf u$, and from Lemma 2 we have $\inf IS_\epsilon u = \inf u$. If $[u < \lambda]$ contains a neighborhood of infinity, then $[IS_\epsilon u < \lambda] = T'_\epsilon[u < \lambda] = \cup\{X : X \in \mathrm{CC}([u < \lambda]), |X| > \epsilon\}$ is also a neighborhood of infinity and, therefore, $[IS_\epsilon u \geq \lambda]$ is compact. Note that this holds for all $\lambda > \inf u$.

Finally, since $u \leq IS_\epsilon u$, we have that $[u = \inf u] \subseteq [IS_\epsilon u \geq \inf u]$. We deduce that, if $[u = \inf u]$ contains a neighborhood of infinity, then $[IS_\epsilon u = \inf u]$ contains a neighborhood of infinity. Now, if $[u = \inf u] = \emptyset$, then $\cup_{\lambda > \inf u}[u \geq \lambda] = D$, and, since $IS_\epsilon u \geq u$, also $\cup_{\lambda > \inf u}[IS_\epsilon u \geq \lambda] = D$. We have that $[IS_\epsilon u = \inf u] = \emptyset$. □

Proposition 11. *Suppose that D is locally compact, locally connected and the measure of its open sets is > 0. Assume also that D contains at most a finite number of connected components of area $\geq \epsilon$. Let $u\colon D \to \mathbb{R}$ be a continuous function. Suppose that $[u \geq \lambda]$ are compact for any $\lambda > \inf u$. Then $SI^\epsilon u$ is continuous. The upper level sets $[SI^\epsilon u \geq \lambda]$ are compact for any $\lambda > \inf u$. Moreover, if $[u = \inf u]$ contains a neighborhood of infinity, the same can be said for $[SI^\epsilon u = \inf u]$.*

Remark 4. If D is compact or $\lim_{|x| \to \infty} u(x) = -\infty$, then $[u \geq \lambda]$ is compact for all $\lambda \in \mathbb{R}$.

Proof: We have proved in Proposition 8 that $SI^\epsilon u$ is upper semicontinuous. As in last proposition, to prove that $SI^\epsilon u$ is lower semicontinuous it is sufficient to prove that $[SI^\epsilon u \leq \lambda]$ are closed sets for all $\lambda > \inf u$.

Let $\lambda > \inf u$. Then, by Proposition 7, i), we have

$$\begin{aligned}[SI^\epsilon u \le \lambda] &= \cap_{\delta>0}[SI^\epsilon u < \lambda+\delta] \\ &= \cap_{\delta>0}(\mathbb{R}^N \setminus [SI^\epsilon u \ge \lambda+\delta]) \\ &= \cap_{\delta>0}(\mathbb{R}^N \setminus T_\epsilon[u \ge \lambda+\delta]) \\ &= \mathbb{R}^N \setminus \cup_{\delta>0} \cup \{X : X \in \mathrm{CC}([u \ge \lambda+\delta]),\, |X| \ge \epsilon\}.\end{aligned}$$

We claim that

$$(10)\quad \cup_{\delta>0} \cup\{X : X \in \mathrm{CC}([u \ge \lambda+\delta]),\, |X| \ge \epsilon\} = \cup\{Y : Y \in \mathrm{CC}([u > \lambda]),\, |Y| > \epsilon\}.$$

Let $X \in \mathrm{CC}([u \ge \lambda+\delta])$ be such that $|X| \ge \epsilon$ for some $\delta > 0$. Let $Y = \mathrm{cc}([u > \lambda], X)$. Since Y and X are, respectively, open and closed in D, we have that $|Y| > |X| \ge \epsilon$. This proves that

$$(11)\quad \cup_{\delta>0} \cup\{X : X \in \mathrm{CC}([u \ge \lambda+\delta]),\, |X| \ge \epsilon\} \subseteq \cup\{Y : Y \in \mathrm{CC}([u > \lambda]),\, |Y| > \epsilon\}.$$

Now, let $Y \in \mathrm{CC}([u > \lambda])$ be such that $|Y| > \epsilon$. Let $p \in Y$ so that $Y = \mathrm{cc}([u > \lambda], p)$. As in Proposition 5, ii), we prove that

$$\mathrm{cc}([u > \lambda], p) = \cup_{\delta>0} \mathrm{cc}([u \ge \lambda+\delta], p).$$

Thus, $\epsilon < |Y| \le \lim_{\delta\to 0} |\mathrm{cc}([u \ge \lambda+\delta], p)|$. We conclude that there is some $\delta > 0$ such that, if $X = \mathrm{cc}([u \ge \lambda+\delta], p)$ then $|X| \ge \epsilon$. We have proved that

$$\cup\{Y : Y \in \mathrm{CC}([u > \lambda]),\, |Y| > \epsilon\} \subseteq \cup_{\delta>0} \cup \{X : X \in \mathrm{CC}([u \ge \lambda+\delta]),\, |X| \ge \epsilon\}.$$

The identity in (10) is proved. Since the right hand side of this identity is an open set in D, whose complement in D is $[SI^\epsilon u \le \lambda]$ we deduce that the last set is closed in D. We have proved that $SI^\epsilon u$ is continuous.

Since $[SI^\epsilon u \ge \lambda] = T_\epsilon[u \ge \lambda]$ we deduce that the sets $[SI^\epsilon u \ge \lambda]$ are compact for any $\lambda > \inf u$. The rest of the statement follows from Proposition 8. □

Combining Propositions 10 and 11 we obtain

Proposition 12. *Suppose that D is closed, locally connected and the measure of its open sets is > 0. We also suppose that D is either compact or it contains a neighborhood of infinity. Let $u\colon D \to \mathbb{R}$ be a continuous function such that $[u \ge \lambda]$ is compact for any $\lambda > \inf u$. Then $IS_\epsilon SI^\epsilon u$ and $SI^\epsilon IS_\epsilon u$ are continuous.*

Remark 5. Motivated by the study of a family of filters by reconstruction ([**23**], [**24**], [**37**], [**43**], [**44**]), J. Serra and P. Salembier ([**42**], [**39**]) introduced the notion of connected operators. Such operators simplify the topographic map of the image. These filters have become very popular because, on an experimental basis, they have been claimed to simplify the image while preserving contours. This property has made them very attractive for a large number of applications such as noise cancellation or segmentation ([**27**], [**45**]). More recently, they have become the basis of a morphological approach to image and video compression (see [**38**] and references therein, and more recently [**16**]).

4. Structure of the simplified topographic map

Assuming that the connected components of the upper and lower level sets are of certain size, then there is only a finite number of maximal monotone sections in the topographic map of u. The idea being that given two of them there exists a set of area $\geq \delta$ contained between them.

Let $\overline{\Omega}$ be a Jordan domain in $\mathbb{R}^N$, i.e., the closure of the bounded connected component of the complement of a subset of $\mathbb{R}^N$ homeomorphic to S^{N-1}, the sphere of $\mathbb{R}^N$. Let Ω be the interior of $\overline{\Omega}$. Note that, in particular, $\overline{\Omega}$ is compact, locally connected and the measure of its open sets is > 0.

A word of caution: when we say that, if X is a connected component of an upper or lower level set, then $|X| \geq \delta$, we mean $X = \mathrm{cc}([u \geq \lambda], p)$ or $X = \mathrm{cc}([u < \lambda], p)$ for some $\lambda \in \mathbb{R}$, $p \in \overline{\Omega}$.

Our purpose is to prove the following results.

Theorem 1. *Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a continuous function. Assume that there is some $\delta > 0$ such that, if X is a connected component of an upper or lower level set, then $|X| \geq \delta$. Then there is a finite number of maximal monotone sections in the topographic map of u.*

To prove this theorem we shall first prove:

Proposition 13. *Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a continuous function. Assume that there is some $\delta > 0$ such that, if X is a connected component of an upper or lower level set, then $|X| \geq \delta$. Then, for each $\lambda, \mu \in \mathbb{R}$ with $\lambda \leq \mu$, the set $U_{\lambda,\mu}$ has a finite number of connected components.*

According to Section 2 the main implication of this proposition is that we may define the maximal monotone sections containing a given point. On the other hand, if u is continuous, then by the results of Section 3, the function $\overline{u} = IS_\delta SI^\delta u$ is also continuous and Proposition 3 holds, namely, if X is a connected component of an upper or lower level set, then

$|X| \geq \delta$. One can also take $\overline{u} = SI^{\delta}IS_{\delta}u$. In any case, if a continuous function u does not satisfy the assumption of Proposition 13, then by filtering the connected components of size less than δ, these assumption are satisfied. To prove Proposition 13 let us recall some topological facts.

4.1. Some topological preliminaries.

Definition 3. Let $A \subseteq \overline{\Omega}$. We call holes of A in $\overline{\Omega}$ the components of $\overline{\Omega} \setminus A$. Given a hole T of A in $\overline{\Omega}$, we call saturation of A with respect to T the set $\overline{\Omega} \setminus T$ and denote it by $\mathrm{sat}(A, T)$. We shall refer to T as the external hole of A and the other holes of A as the internal holes of A. If $p \in \overline{\Omega} \setminus A$, and T is the hole of A containing p, we define $\mathrm{sat}(A, p) = \mathrm{sat}(A, T)$. Note that $\mathrm{sat}(A, T)$ is the union of A and its internal holes. We call saturated sets associated to A the family of saturations of A with respect to its holes.

If no confusion arises, we shall write $\mathrm{sat}(A)$ instead of $\mathrm{sat}(A, T)$. Thus $\mathrm{sat}(A)$ denotes a saturated set containing A chosen from the family of all saturations of A. We shall also speak of holes of A instead of holes of A in $\overline{\Omega}$. Let A be a bounded subset of $\mathbb{R}^N$. If we choose as external hole of A in $\mathbb{R}^N$ the unbounded component of $\mathbb{R}^N \setminus A$, then the internal holes of A are the bounded connected components of $\mathbb{R}^N \setminus A$.

Definition 4. A sequence $A_1, \dots, A_p$ of subsets of $\overline{\Omega}$ is called a chain if each A_i is contained in a hole of A_{i-1}, $i = 2, \dots, p$.

Definition 5 ([**22**, vol. II, p. 104])**.** A topological space Z is said to be unicoherent if it is connected and for any two closed connected sets A, B in Z such that $Z = A \cup B$ we have that $A \cap B$ is connected.

Lemma 3. *Let $A \subseteq \overline{\Omega}$. If A is open (resp. closed) in $\overline{\Omega}$, then its saturated sets are open (resp. closed) in $\overline{\Omega}$.*

Proof: If A is open in $\overline{\Omega}$, $\overline{\Omega} \setminus \mathrm{sat}(A)$, being a connected component of a closed set in $\overline{\Omega}$, it is also closed in $\overline{\Omega}$. Thus $\mathrm{sat}(A)$ is open in $\overline{\Omega}$. If A is closed in $\overline{\Omega}$, $\overline{\Omega} \setminus \mathrm{sat}(A)$, being a connected component of an open set in $\overline{\Omega}$, it is also open in $\overline{\Omega}$. Thus $\mathrm{sat}(A)$ is closed in $\overline{\Omega}$. □

Lemma 4. i) *The saturation is a monotonous operation, i.e., if $A \subseteq B$ and $p \in \overline{\Omega} \setminus B$, then $\mathrm{sat}(A, p) \subseteq \mathrm{sat}(B, p)$.*

ii) *Let K_n be a decreasing sequence of continua, $K = \cap_n K_n$, and $p \notin K$. Then $\mathrm{sat}(K, p) = \cap_n \mathrm{sat}(K_n, p)$.*

Proof: The proof of i) is immediate and we shall skip it. Let us prove ii). Obviously, for n large enough, $p \notin K_n$. By the monotonicity of the saturation, $\operatorname{sat}(K,p) \subseteq \cap_n \operatorname{sat}(K_n,p)$. Let $q \notin \operatorname{sat}(K,p)$, $q \neq p$, and let γ be an arc joining p to q such that $\operatorname{Im}(\gamma) \cap \operatorname{sat}(K,p) = \emptyset$. Then, for n large enough, we have $\operatorname{Im}(\gamma) \cap K_n = \emptyset$. Hence, both p and q are in the same hole of K_n. Hence, $\operatorname{Im}(\gamma) \cap \operatorname{sat}(K_n,p) = \emptyset$, and, in particular, $q \notin \operatorname{sat}(K_n,p)$, for n large enough. Thus, $q \notin \cap_n \operatorname{sat}(K_n,p)$. We conclude that $\cap_n \operatorname{sat}(K_n,p) \subseteq \operatorname{sat}(K,p)$.

Lemma 5. *Let $u \in C(\overline{\Omega})$. We assume that there is some $\delta > 0$ such that, if K is a connected component of an upper or lower level set, then $|K| \geq \delta$. Then for each $\lambda \in \mathbb{R}$ there is a finite number of connected components of $[u \geq \lambda]$ and each component has a finite number of holes.*

Proof: Let $\lambda \in \mathbb{R}$. Since each connected component of $[u \geq \lambda]$ has area $\geq \delta$, there must be a finite number of them. Let Y be a component of $[u \geq \lambda]$ and let H be a hole of Y. Since H is open, H cannot be covered by components of $[u \geq \lambda]$. Hence $H \cap [u < \lambda] \neq \emptyset$. We conclude that each hole of Y contains a component of $[u < \lambda]$. Hence there may be only a finite number of them.

Lemma 6. *Let $u \in C(\overline{\Omega})$. Let X be a connected component of $[\lambda \leq u \leq \mu]$, $\lambda < \mu$, and let L be a hole of X. Assume that $\partial L \subseteq \partial[u < \lambda]$. Then $\partial L \subseteq \partial(L \cap [u < \lambda])$.*

Proof: Let $p \in \partial L$. Since $B(p,r) \cap (\overline{\Omega} \setminus L) \neq \emptyset$ for all $r > 0$, we also have that $B(p,r) \cap (\overline{\Omega} \setminus (L \cap [u < \lambda])) \neq \emptyset$ for all $r > 0$. Since $p \in \partial L \subseteq \partial[u < \lambda]$, we know that $B(p,r) \cap L \neq \emptyset$, $B(p,r) \cap [u < \lambda] \neq \emptyset$ for all $r > 0$. Let us prove that $B(p,r) \cap L \cap [u < \lambda] \neq \emptyset$ for all $r > 0$. Let $r > 0$ and suppose that $B(p,r) \cap L \subseteq [u \geq \lambda]$. Since $\lambda < \mu$, by taking r smaller, if necessary, we may assume that $B(p,r) \cap L \subseteq [\lambda \leq u \leq \mu]$. Let A_i, $i \in I$, be the family of (open) connected components of $B(p,r) \cap L$. Since $\overline{\Omega}$ is locally connected, then $\partial A_i \subseteq \partial(B(p,r) \cap L)$ ([**22**, vol. II, p. 169, 3]). Hence, $\partial A_i \subseteq (\partial B(p,r) \cap L) \cup \partial L$. If $\partial A_i \cap \partial L \neq \emptyset$, then $\overline{A_i} \cap X \neq \emptyset$ because $\partial L \subseteq X$. Since $\overline{A_i}$ is connected and contained in $[\lambda \leq u \leq \mu]$, then $\overline{A_i} \subseteq X$, a contradiction. Thus, $\partial A_i \cap \partial L = \emptyset$, and, therefore, $\partial A_i \subseteq \partial B(p,r)$. We may assume this to be true for all $i \in I$. It follows that $\partial(B(p,r) \cap L) = \partial(\cup_i A_i) \subseteq \overline{\cup_i \partial A_i} \subseteq \partial B(p,r)$ ([**22**, vol. II, p. 169, 1]). Since $p \in \partial L$, we have that $p \in \partial(B(p,r) \cap L)$, and we would have that $p \in \partial B(p,r)$. This contradiction proves that $B(p,r) \cap L \cap [u < \lambda] \neq \emptyset$ for all $r > 0$. We conclude that $p \in \partial(L \cap [u < \lambda])$. □

Proposition 14. *Let $u \in C(\overline{\Omega})$. Assume that there is some $\delta > 0$ such that, if K is a connected component of an upper or lower level set of u, then $|K| \geq \delta$. Let X be a connected component of $[\lambda \leq u \leq \mu]$, $\lambda \leq \mu$, and let L be a hole of X. Then there is some $\eta > 0$ such that either*

i) $\mathrm{sat}(X, L) = \mathrm{sat}(\mathrm{cc}([u \geq \lambda], X), L)$, *and $u < \lambda$ on $L_\eta := \{p \in L : d(p, X) < \eta\}$, or*

ii) $\mathrm{sat}(X, L) = \mathrm{sat}(\mathrm{cc}([u \leq \mu], X), L)$, *and $u > \mu$ on $L_\eta := \{p \in L : d(p, X) < \eta\}$.*

Remark 6. i) The assertions concerning the saturations in the above statement are true without the assumption that the connected components of upper or lower level sets have area $\geq \delta$, and it can be proved by approximation of u by $IS_\epsilon SI^\epsilon u$.

ii) The set L_η will be called a band around X. If a hole is an external hole, we call L_η the external band around X. If a hole L is an internal hole, we call L_η an internal band around X.

Proof: Let H be any hole of X. Since $\overline{\Omega}$ is unicoherent, $\mathrm{sat}(X, H)$, $\overline{H}$ are closed connected sets, and $\mathrm{sat}(X, H) \cup \overline{H} = \overline{\Omega}$, we have that $\partial H = \mathrm{sat}(X, H) \cap \overline{H}$ is connected. We observe that $\partial H \subseteq \partial[u < \lambda] \cup \partial[u > \mu]$. Indeed, since $\overline{\Omega}$ is locally connected, $\partial X \subseteq \partial[\lambda \leq u \leq \mu]$ ([**22**, vol. II, p. 169]) and

$$\begin{aligned} \partial H \subseteq \partial X &\subseteq \partial[\lambda \leq u \leq \mu] \\ &= \partial([u < \lambda] \cup [u > \mu]) \\ &= \partial[u < \lambda] \cup \partial[u > \mu]. \end{aligned}$$

Moreover, if $\lambda < \mu$, $\partial[u < \lambda] \cap \partial[u > \mu] = \emptyset$ and, being ∂H a connected set, we have either $\partial H \subseteq \partial[u < \lambda]$ or $\partial H \subseteq \partial[u > \mu]$. Suppose first that $\lambda < \mu$. Without loss of generality, we may assume that $\partial L \subseteq \partial[u < \lambda]$. Our purpose is to prove that L is also a hole of $\mathrm{cc}([u \geq \lambda], X)$, hence $\mathrm{sat}(X, L) = \mathrm{sat}(\mathrm{cc}([u \geq \lambda], X), L)$. For that, we shall prove that, for some $\eta > 0$, $u < \lambda$ on $\{p \in L : d(p, X) < \eta\}$.

The result is obviously true if $L = \emptyset$ or $L \subseteq [u < \lambda]$. Thus we may assume that $L \cap [u \geq \lambda] \neq \emptyset$. By Lemma 5, there is a finite number of connected components of $[u \geq \lambda]$ intersecting L, and each of them has, at most, a finite number of holes. Let Y be a component of $[u \geq \lambda]$ intersecting L. Observe that, by last lemma, $[u < \lambda] \neq \emptyset$. Hence $Y \neq \overline{\Omega}$ and it has some hole. If all holes of Y are contained in $\mathrm{sat}(X, L)$, then $L \subseteq Y$. In that case, $\emptyset \neq L \cap [u < \lambda] \subseteq Y \cap [u < \lambda] = \emptyset$, and we obtain a contradiction. Thus, there are holes of Y intersecting L.

Let W be a hole of Y such that $W \cap L \neq \emptyset$. We claim that $\partial W \cap \partial L = \emptyset$. Indeed, let $p \in Y \cap L$, $q \in W \cap L$. Let γ be an arc joining p and q whose image is contained in L. Then γ intersects ∂W at some point $x \in L$. As in the above proof, ∂W is a connected set contained in $[u = \lambda]$. Hence, if $\partial W \cap \partial L \neq \emptyset$, since $\partial L \subseteq X$, then X would contain ∂W, hence, also $x \in \partial W \cap L$, a contradiction. Our claim follows.

We claim that $\partial Y \cap \partial L = \emptyset$. Suppose, by contradiction that $\partial Y \cap \partial L \neq \emptyset$. Since $\partial L \subseteq [u = \lambda]$, then $\partial L \subseteq Y$. Since $Y \cap W = \emptyset$, we also have that $\partial L \cap W = \emptyset$. Therefore $\partial L \cap \overline{W} = \emptyset$. Let W_i, $i = 1, \ldots, p$, be the family of holes of Y intersecting L. We have just proved that $\mathrm{dist}(\partial L, \overline{W_i}) \geq \eta > 0$ for some $\eta > 0$ and all $i = 1, \ldots, p$. Let $p \in \partial L$ and let $r < \eta$ be such that $B(p, r) \cap \overline{W_i} = \emptyset$, $i = 1, \ldots, p$. Under these circumstances, it is immediate to see that

$$B(p, r) \cap L = B(p, r) \cap L \cap Y.$$

Then

$$\emptyset \neq B(p, r) \cap L \cap [u < \lambda] = B(p, r) \cap L \cap Y \cap [u < \lambda] = \emptyset,$$

the left hand side being nonempty because $p \in \partial(L \cap [u < \lambda])$. This contradiction proves that $\partial Y \cap \partial L = \emptyset$.

Thus, there is some $r > 0$ such that either $\partial L + B(0, r) \subseteq Y$, or $(\partial L + B(0, r)) \cap Y = \emptyset$. In the first case, we conclude that, by choosing $r > 0$ small enough, $\partial L + B(0, r) \subseteq [\lambda \leq u \leq \mu]$, hence, also $\partial L + B(0, r) \subseteq X$, a contradiction. Thus, we may assume that $(\partial L + B(0, r)) \cap Y = \emptyset$ for all connected components Y of $[u \geq \lambda]$ intersecting L. Then, $\mathrm{dist}([u \geq \lambda] \cap L, \partial L) > 0$, and, therefore, for some $\eta > 0$, $u < \lambda$ on $\{p \in L : d(p, X) < \eta\}$. This implies that L is a hole of $\mathrm{cc}([u \geq \lambda], X)$, and $\mathrm{sat}(X, L) = \mathrm{sat}(\mathrm{cc}([u \geq \lambda], X), L)$.

Let us consider the case $\lambda = \mu$. Let X be a connected component of $[u = \lambda]$ and $y \in X$. Then $X = \cap_n X_n$ where $X_n = \mathrm{cc}([\lambda \leq u \leq \lambda + \frac{1}{n}], y)$. Let $p \in L$. Then, by Lemma 4, we know that $\mathrm{sat}(X, p) = \cap_n \mathrm{sat}(X_n, p)$. Without loss of generality, we may assume that $p \notin X_n$ for all $n \geq 1$. But, according to the first part of the proof, we have that either $\mathrm{sat}(X_n, p) = \mathrm{sat}(\mathrm{cc}([u \geq \lambda], y), p)$, or $\mathrm{sat}(X_n, p) = \mathrm{sat}(\mathrm{cc}([u \leq \lambda + \frac{1}{n}], y), p)$. In the first case, we conclude that $\mathrm{sat}(X, p) = \mathrm{sat}(\mathrm{cc}([u \geq \lambda], y), p)$. In the second case, using again Lemma 4, ii), we have that $\cap_n \mathrm{sat}(\mathrm{cc}([u \leq \lambda + \frac{1}{n}], y), p) = \mathrm{sat}(\mathrm{cc}([u \leq \lambda], y), p)$. Hence, $\mathrm{sat}(X, p) = \mathrm{sat}(\mathrm{cc}([u \leq \lambda], y), p)$.

When $\operatorname{sat}(X,p) = \operatorname{sat}(\operatorname{cc}([u \geq \lambda], y), p)$, L is a hole of $\operatorname{cc}([u \geq \lambda], y)$. Hence $\partial L \subseteq \partial[u < \lambda]$ and the argument above proves that there is some $\eta > 0$ such that $u < \lambda$ on $L_\eta = \{p \in L : d(p,X) < \eta\}$. When $\operatorname{sat}(X,p) = \operatorname{sat}(\operatorname{cc}([u \leq \lambda], y), p)$, L is a hole of $\operatorname{cc}([u \leq \lambda], y)$. Then $\partial L \subseteq \partial[u > \lambda]$ and again the previous argument proves that there is some $\eta > 0$ such that $u > \lambda$ on $L_\eta = \{p \in L : d(p,X) < \eta\}$. □

4.2. The number of components of $U_{\lambda,\mu}$ is finite.

Lemma 7. *Let $u \in C(\overline{\Omega})$. Assume that there is some $\delta > 0$ such that, if K is a connected component of an upper or lower level set, then $|K| \geq \delta$. Let $\lambda \leq \mu$.*

i) *Let X be a connected component of $U_{\lambda,\mu}$. Then $|\operatorname{sat}(X)| \geq \delta$.*
ii) *Let X, Y be two connected components of $U_{\lambda,\mu}$ with X contained in a hole of Y. Suppose that both satisfy the same alternative in Proposition 14. Then $|\operatorname{sat}(Y) \setminus \operatorname{sat}(X)| \geq \delta$.*

Proof: i) is a consequence of Proposition 14. To prove ii), to fix ideas, let us assume that there is some $\eta > 0$ such that $u < \lambda$ on $\{p \in L_X : d(p,X) < \eta\}$ and on $\{p \in L_Y : d(p,Y) < \eta\}$, where L_X and L_Y are the external holes of X, Y, respectively. Then, there is a connected component of $[u \geq \lambda]$ contained in $\operatorname{sat}(Y) \setminus \operatorname{sat}(X)$, and, therefore, $|\operatorname{sat}(Y) \setminus \operatorname{sat}(X)| \geq \delta$. □

Lemma 8. *Let $u \in C(\overline{\Omega})$. Assume that there is some $\delta > 0$ such that if X is a component of an upper or lower level set, then $|X| \geq \delta$. Let $J \subseteq \mathbb{N}$. Let K_j, $j \in J$, be a connected component of U_{α_j,β_j}, $\alpha_j \leq \beta_j$, such that $K_i \cap K_j = \emptyset$ for all $i \neq j$, $i,j \in J$.*

i) *For each $j \in J$ let T_i^j, $i = 1,\dots,h_j$, be the holes K_j containing some K_i, $i \in J$. Then*

$$|\cup_{i=1}^{h_j} T_i^j| \geq h_j \delta.$$

In particular

$$\sup_j h_j \leq \frac{|\Omega|}{\delta}.$$

ii) *Suppose that J is countable. Then there is an infinite chain formed by sets of the family K_j.*

Proof: i) Suppose that T_i^j contains $K_{n_i^j}$. Then, it contains also a saturation of $K_{n_i^j}$. Thus, by Lemma 7, $|\operatorname{sat}(K_{n_i^j})| \geq \delta$. This implies the statement of the present lemma.

ii) Let $j \in J$. Observe that the number of K_i in the holes (i.e., the components of its complement) of K_j is infinite. By i), there is a hole of K_j which contains an infinite number of K_i. Let T^1 be this hole. Suppose that there is an infinity of K_i, K_{i_r}, in T^1 such that $\mathrm{sat}(K_{i_r})$ are two by two disjoint. Using Lemma 7, i), we obtain that

$$|T^1| \geq k\delta$$

for all $k \geq 1$. Thus there is only a finite system of components K_i such that $\mathrm{sat}(K_i)$ are two by two disjoint. Then one of them, say K_{i_1}, contains an infinity of $K_i's$ in his system of holes. Again, by the previous argument, there is a hole of K_{i_1}, say T^2, containing an infinite number of K_i. Repeating the same argument we find a subsequence K_{i_n}, $n = 1, 2, \dots$, such that $K_{i_{n+1}}$ is contained in a hole of K_{i_n}. □

Proof of Proposition 13: Suppose that $U_{\lambda,\mu}$ contains a countable family of connected components K_n. By Lemma 8, ii), there is an infinite chain formed by sets of the family K_j. By extracting a subsequence of K_j, if necessary, we may assume that they satisfy the same alternative given in Proposition 14. Now, using Lemma 7, ii), we obtain an infinite area in this chain. This contradiction proves that there is only a finite number of connected components of $U_{\lambda,\mu}$. □

4.3. The number of maximal monotone sections is finite.

Theorem 2. *Let $u \in C(\overline{\Omega})$. Assume that there is some $\delta > 0$ such that, if X is a connected component of an upper or lower level set then, $|X| \geq \delta$. Then there is a finite number of maximal monotone sections in the topographic map of u.*

To proceed with the proof, assume that there is a sequence $\{S_i\}_{i=1}^{\infty}$ of maximal monotone sections, each one associated with an interval (open, closed, halfopen or halfclosed) which will be denoted by $I_i = \{a_i, b_i\}$, $a_i \leq b_i$. Let $z_i = \frac{a_i+b_i}{2}$. Since u is bounded, passing to a subsequence, if necessary, we may assume that $z_i \to z$ as $i \to \infty$. Then, modulo a subsequence, we may assume that the intervals I_i intersect each other or they are two by two disjoint. Indeed, if the number of indexes $i \in \mathbb{N}$ such that $a_i = b_i$ is infinite we have a sequence of intervals I_i which intersect each other (in case $a_i = b_i = z$) or which are two by two disjoint. Thus we may assume that $a_i < b_i$, for all $i \geq 1$. Again, taking a subsequence if necessary, we may assume that a_i, b_i are monotone sequences. Let $a = \lim_i a_i$, $b = \lim_i b_i$. Note that $z = \frac{a+b}{2}$. The following cases are possible:

i) $a_i \searrow$, $b_i \searrow$: If $a < z$, then also $z < b$ and $(a_i, z] \subseteq \{a_i, b_i\}$. The intervals I_i intersect each other (even have a common intersection point). If $a = z$, then also $b = z$, and there are two possibilities: $(p1)$ either $a_i = z$ for all i, or $(p2)$ modulo a subsequence, we have $z < a_i$ for all i. In the first case, $I_i = \{z, b_i\}$ and these intervals intersect each other. In the second case, by taking a subsequence, if necessary, we may assume that $b_{i+1} < a_i < b_i$.

ii) $a_i \nearrow$, $b_i \searrow$: The intervals I_i form a decreasing sequence, thus, intersecting each other.

iii) $a_i \searrow$, $b_i \nearrow$: The sequence of intervals I_i is monotone increasing, thus, having a common intersection.

iv) $a_i \nearrow$, $b_i \nearrow$: If $a < z$, then for i large enough $a < b_i$ and $(a, b_i) \subseteq I_i$. Modulo a subsequence, the intervals I_i have a common intersection point. If $a = z$, then also $b = z$, and there are two possibilities: $(p3)$ either $b_i = z$ for all i, or, $(p4)$ modulo a subsequence, $b_i < z$ for all i. In the first case the sequence $I_i = \{a_i, z\}$, with $a_i \nearrow$, and we conclude that any finite system of intervals has a nonempty intersection. In the second case, modulo a subsequence, we may assume that $a_i < b_i < a_{i+1}$ for all i, i.e., the intervals I_i are two by two disjoint.

By Proposition 1, we have that $S_i \cap S_j = \emptyset$ for all $i \neq j$. Let S_i' be a connected component of U_{α_i,β_i} contained in S_i, with $[\alpha_i, \beta_i] \subseteq I_i$. Then, by Lemma 8, there is an infinite chain made of sets S_i'. Thus we may assume that S_i' form a chain. In this case also S_i forms a chain. We are going to exclude this case. First, we consider the case: $I_i \cap I_j \neq \emptyset$ for all $i, j \in \mathbb{N}$.

Lemma 9. *Let $u \in C(\overline{\Omega})$ be such that each section $U_{\alpha,\beta}$ has a finite number of connected components. Suppose that $[a, b] \cap [c, d] \neq \emptyset$. Let $Q_{a,b}$, $Q_{c,d}$ be connected components of $U_{a,b}$, $U_{c,d}$, respectively. Suppose that $T_{a,b}$, $T_{c,d}$ are holes of $Q_{a,b}$, $Q_{c,d}$, respectively, and $\operatorname{sat}(Q_{c,d})$ is contained in $T_{a,b}$. Then $|\operatorname{sat}(Q_{a,b}) \setminus T_{c,d}| \geq \delta$.*

Proof: Let W_0, W_1 (resp. W_2, W_3) be external and internal bands of $Q_{a,b}$ (resp. $Q_{c,d}$) such that $W_3 \subseteq T_{c,d}$, $W_2, W_1 \subseteq T_{a,b}$ with $W_2 \subseteq \operatorname{sat}(W_1)$ which exist by Proposition 14. To simplify our discussion we shall say that W is higher than $k \in \mathbb{R}$ (resp. lower than $k \in \mathbb{R}$) if $u(z) > k$ ($u(z) < k$) for all $z \in W$, where W denotes any of the bands W_i, $i = 0, 1, 2, 3$. Note that, by Lemma 14, W_0 and W_1 are either higher than b or lower than a, and W_2, W_3 are higher than d or lower than c.

Case $a \leq c$. Since $u(z) \geq a$ for all $z \in Q_{a,b} \cup Q_{c,d}$, if W_1 is lower than a, then there is a connected component of $[u < a]$ in $T_{a,b} \setminus \mathrm{sat}(Q_{c,d})$. Suppose that W_1 is higher than b. If W_0 is higher than b, then there is a connected component of $[u \leq b]$ contained in $Q_{a,b}$. Assume that W_0 is lower than a. If W_2, W_3 are higher than d, with a similar argument we have that there is a connected component of $[u \leq d]$ contained in $Q_{c,d}$. If W_2 is higher than d and W_3 lower than c, then there is a connected component of $[u > d]$ in $\mathrm{sat}(Q_{a,b}) \setminus T_{c,d}$. If W_2 is lower than c, since $u \geq c$ on $Q_{c,d}$, then there is a connected component of $[u < c]$ contained in $T_{a,b} \setminus \mathrm{sat}(Q_{c,d})$.

Case $c \leq a$. Suppose that W_1 is lower than a. Since $u(z) \geq a$ for all $z \in Q_{a,b}$ and $u(z) \geq c$ for all $z \in Q_{c,d}$, if W_2 is higher than d, then there is a connected component of $[u < a]$ in $T_{a,b} \setminus \mathrm{sat}(Q_{c,d})$. If W_2 is lower than c, then there is a connected component of $[u < c]$ in $T_{a,b} \setminus \mathrm{sat}(Q_{c,d})$. If W_1 is higher than b we argue as in the previous case. □

Lemma 10. *Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of different maximal monotone sections, each one associated with an interval I_i. Suppose $I_i \cap I_j \neq \emptyset$ for all $i, j \in J$, $i \neq j$. Then, there is no infinite chain formed with this sets.*

Proof: Modulo a subsequence, we may assume that S_i is an infinite chain. By Proposition 1, we know that $S_i \cap S_j = \emptyset$ for all $i \neq j$. From the assumption, it follows that given $N \geq 1$, there are closed subintervals $I_i^* \subseteq I_i$, $1 \leq i \leq N$, such that $I_i^* \cap I_j^* \neq \emptyset$ for all $i, j \in \{1, \dots, N\}$, $i \neq j$. Let $S_i^* \subseteq S_i$ be a connected component associated with the closed interval I_i^*, $i \leq N$. To fix ideas, since the union of S_i^* does not cover $\overline{\Omega}$, we may choose a point $p \in \overline{\Omega} \setminus \cup_{i=1}^{N} S_i^*$ and always consider the component of $\overline{\Omega} \setminus S_i^*$ containing p as the external component of S_i^*. Let k be the number disjoint saturations of the sets S_i^*. Then there is a chain of length at least $\left[\frac{N}{k}\right]$ where $[\lambda]$ denotes the integer part of λ. Then, we have $| \cup_{i=1}^{N} \mathrm{sat}(S_i^*)| \geq k\delta$ and, by last lemma, $| \cup_{i=1}^{N} \mathrm{sat}(S_i^*)| \geq \frac{1}{2} \left[\frac{N}{k}\right] \delta$. Observe that either k or $\frac{N}{k}$ goes to infinity with N. In any case, Ω would have unbounded measure. □

At several places we shall use inversions in $\mathbb{R}^N$ with respect to some point $p \in \mathbb{R}^N$. We call inversion in $\mathbb{R}^N$ with respect to p any map $\varphi_p(x) = R^2 \frac{x-p}{|x-p|^2}$, $R > 0$. It maps $B(p, R)$ bijectively into $\mathbb{R}^N \setminus \overline{B(p, R)}$ and it keeps the sphere or radius R around p fixed.

Lemma 11. *Let S_i be a sequence of maximal monotone sections associated with $I_i = \{a_i, b_i\}$, $i \in \mathbb{N}$. Assume that $a_i \leq b_i < a_{i+1}$, or $b_{i+1} < a_i \leq b_i$ for all i. Then there is no infinite chain made of sets S_i.*

Proof: Suppose that the sequence S_i forms a chain. By eliminating one of the S_i, if necessary, we may assume that the union of the S_i does not cover $\overline{\Omega}$ and fix a point p outside all sets $\mathrm{sat}(S_i)$ which contain p in its external component. By Lemma 8, we may assume that there exist closed monotone sections Q_i contained in S_i, associated to the intervals $[c_i, d_i] \subseteq \{a_i, b_i\}$, and forming a chain. Let T_i be the hole of Q_i containing Q_{i+1}. We shall prove that

$$|\,\mathrm{sat}(Q_i) \setminus T_{i+1}| \geq \delta, \tag{12}$$

for all $i \geq 1$. It suffices to consider the case $a_i \leq b_i < a_{i+1}$ for all i. Indeed, in case we have that $b_{i+1} < a_i \leq b_i$ for all i, for each i fixed, for instance $i = 1$, we may consider a point $q \in T_2$ and make an inversion φ with respect to the point q, to end up with concatenated sets $\varphi(Q_2)$, $\varphi(Q_1)$, exterior and interior, respectively, associated with intervals $[c_2, d_2]$, $[c_1, d_1]$, satisfying $d_2 < c_1$. In what follows we shall prove that under these assumptions $\mathrm{sat}(\varphi(Q_2)) \setminus \varphi(\overline{\Omega} \setminus \mathrm{sat}(Q_1))$ contains an upper or lower level set of $u \circ \varphi$. This implies that $\mathrm{sat}(Q_1) \setminus T_2$ contains an upper or lower level set of u and, as a consequence, $|\,\mathrm{sat}(Q_1) \setminus T_2| \geq \delta$.

Assume that $a_i \leq b_i < a_{i+1}$ for all i. Take $i = 1$. Let W_0, W_1 (resp. W_2, W_3) be external and internal bands of Q_1 (resp. Q_2). Suppose that W_1 is lower than c_1. Since $u(z) \geq c_1$ for all $z \in Q_1$ and $u(z) \geq c_2$ for all $z \in Q_2$, then there is a connected component of $[u < c_1]$ contained in $T_1 \setminus \mathrm{sat}(Q_2)$. Suppose that W_1 is higher that d_1. If W_0 is higher than d_1 then there is a connected component of $[u \leq d_1]$ contained in Q_1. Suppose that W_0 is lower than c_1. If W_3 is lower than c_2, then there is a connected component of $[u \geq c_2]$ contained in $\mathrm{sat}(Q_1) \setminus T_2$. If W_3 and W_2 are higher than d_2 then there is a connected component of $[u \leq d_2]$ contained in Q_2. Thus it remains to consider the case: (cc) W_0 lower than c_1, W_1 higher than d_1, W_2 lower than c_2, W_3 higher than d_2. Suppose that

$$|\,\mathrm{sat}(Q_1) \setminus T_2| < \delta. \tag{13}$$

Let us consider the open set $D = \overline{T_1 \setminus \mathrm{sat}(Q_2)}$. Observe that $\overline{D} = \overline{(\overline{\Omega} \setminus \mathrm{sat}(Q_2))} \cap \overline{T}_1$. Since both sets $\overline{(\overline{\Omega} \setminus \mathrm{sat}(Q_2))}$, $\overline{T}_1$ are closed, connected and their union is $\overline{\Omega}$, then its intersection $\overline{D}$ is connected. Let us prove that $d_1 < u(z) < c_2$ for all $z \in D$. Suppose that there is $p \in D$ such that $u(p) \leq d_1$. Since W_1 is higher than d_1 and $u \geq c_2$ in Q_2 we conclude that there is a connected component of $[u \leq d_1]$ in $T_1 \setminus \mathrm{sat}(Q_2)$. Suppose that there is some $p \in D$ such that $u(p) \geq c_2$. Since W_0 is lower than c_1 and W_2 lower than c_2 we conclude that there is a connected component of $[u \geq c_2]$ in $\mathrm{sat}(Q_1) \setminus \mathrm{sat}(Q_2)$. In both cases we contradict (13).

Since $\overline{D}$ is connected, and $u|_{\partial T_1} = d_1$, $u|_{\partial\,\mathrm{sat}(Q_2)} = c_2$, we have that $\{u(z) : z \in \overline{D}\}$ contains $[d_1, c_2]$. In particular, $\{z \in \overline{D} : \lambda \leq u(z) \leq \mu\}$ is a nonempty set for all $[\lambda, \mu] \subseteq [d_1, c_2]$.

Let $d_1 \leq \lambda \leq \mu \leq c_2$. Let us prove that $X_{\lambda,\mu} := U_{\lambda,\mu} \cap (\mathrm{sat}(Q_1) \setminus T_2)$ is a component of $U_{\lambda,\mu}$. The last paragraph proves that this set is nonempty. Since $u(z) < c_1$ for all $z \in W_0$ and $u(z) > d_2$ for all $z \in W_3$, the set $U_{\lambda,\mu} \cap (\mathrm{sat}(Q_1) \setminus T_2)$ is a union of connected components of $U_{\lambda,\mu}$. Suppose that there are at least two of them H_1, H_2. If the saturation of one of them, say $\mathrm{sat}(H_1)$, does not contain T_2 then $|\mathrm{sat}(Q_1) \setminus T_2| \geq |\mathrm{sat}(H_1)| \geq \delta$. Thus, $\mathrm{sat}(H_1)$, $\mathrm{sat}(H_2)$ contain T_2. We deduce that H_1 and H_2 form a chain. Since both sets are connected components of the same section $U_{\lambda,\mu}$, by Lemma 9, we have that $|\mathrm{sat}(H_1) \setminus T_{H_2}| \geq \delta$ where T_{H_2} is the hole of H_2 containing T_2. In any case, this configuration pays δ, and, therefore also $|\mathrm{sat}(Q_1) \setminus T_2| \geq \delta$. Since this case has been excluded by assumption (13), we conclude that $X_{\lambda,\mu}$ is connected.

Our next claim is that X_{d_1,c_2} is a monotone section. Indeed X_{d_1,c_2} is the only connected component of U_{d_1,c_2} contained in $\mathrm{sat}(Q_1) \setminus T_2$. Now, let $[\hat{\lambda}, \hat{\mu}] \subseteq [d_1, c_2]$. Then

$$\begin{aligned}
\{x \in X_{d_1,c_2} &: \hat{\lambda} \leq u(x) \leq \hat{\mu}\} \\
&= \{x \in \mathrm{sat}(Q_1) \setminus T_2 : d_1 \leq u(x) \leq c_2,\ \hat{\lambda} \leq u(x) \leq \hat{\mu}\} \\
&= \{x \in \mathrm{sat}(Q_1) \setminus T_2 : \hat{\lambda} \leq u(x) \leq \hat{\mu}\} = X_{\hat{\lambda},\hat{\mu}}.
\end{aligned}$$

Since $X_{\hat{\lambda},\hat{\mu}}$ is a connected component of $U_{\hat{\lambda},\hat{\mu}}$, our claim follows.

Let us check that

$$S_1 \cap S_2 \neq \emptyset. \tag{14}$$

This contradiction will prove that, in case (cc), also holds that $|\mathrm{sat}(Q_1) \setminus T_2| \geq \delta$. Thus, the inequality holds in any case. The consequence being that an infinite chain made of sets S_i would imply an infinite area for Ω and Lemma 11 is proved. We observe that $Q_1 \cap X_{d_1,c_2} \neq \emptyset$. Indeed, since W_0 is lower than c_1 and W_1 is higher than d_1, there is a point $\hat{z} \in Q_1$ such that $u(\hat{z}) = d_1$. Thus, $\hat{z} \in Q_1 \cap X_{d_1,c_2}$. Hence

$$X_{d_1,c_2} \subseteq S_1 \tag{15}$$

because S_1 is a maximal monotone section containing Q_1. Now, since W_2 is lower than c_2 and W_3 higher than d_2, there is a point $y \in Q_2$ such that $u(y) = c_2$. Hence

$$X_{d_1,c_2} \cap Q_2 \neq \emptyset. \tag{16}$$

From (16) it follows that

$$X_{d_1,c_2} \subseteq S_2 \tag{17}$$

because S_2 is a maximal monotone section containing Q_2. From (15) and (17) we obtain (14) and the proof of Lemma 11 is completed. □

5. *M*-components versus classical components

As argued in [**7**], the connected components of the level sets of the image contain the information which is invariant by local contrast changes. The geometric description of these basic sets requires a functional model for images. One of the basic models is the so-called BV-model, introduced for the purpose of image denoising by L. I. Rudin, S. Osher and E. Fatemi in [**36**], in which images are functions of bounded variation. The BV-model is a sound model for images which have discontinuities and has become a popular model for image restoration [**36**], [**9**], [**12**], [**46**] and edge detection [**10**]. In [**2**], the authors introduced the WBV-model in which images are functions whose level sets are sets of finite perimeter (modulo a null set of levels). This class of functions contains the functions of bounded variation and is invariant under contrast changes. The notion of connected components can be adapted to sets of finite perimeter in $\mathbb{R}^N$, the so-called M-components introduced in [**2**], and a more precise description of these M-components in terms of Jordan curves was given for $N = 2$ [**2**]. This permited the study of grain filters acting on WBV, in particular, the Vincent-Serra operators SI^ϵ, IS_ϵ described in Section 3 ([**43**], [**44**], [**39**], [**42**]). Grain filters are operators which simplify the structure of connected components of upper and/or lower level sets. S. Masnou ([**26**], [**2**]) studied them as operators in $BV(\Omega)$ proving that they commute with real continuous increasing contrast changes and they decrease total variation. Thus, the WBV-model appears as a suitable functional framework for many problems in image processing. Our purpose now is to prove that for functions in $WBV(\Omega) \cap C(\overline{\Omega})$ the M-components (of positive measure) of almost all its level sets coincide with classical connected components. Thus, in some sense, the M-components are a relaxation of the classical connected components when going from continuous functions to functions in WBV. Let us recall these notions. We shall follow the presentation and the results in [**2**].

5.1. Preliminaries.

We consider a N-dimensional euclidean space $\mathbb{R}^N$, with $N \geq 2$. The Lebesgue measure of a Lebesgue measurable set $E \subseteq \mathbb{R}^N$ will be denoted by $|E|$. For a Lebesgue measurable subset $E \subseteq \mathbb{R}^N$ and a point $x \in \mathbb{R}^N$, the upper and lower densities of E at x are respectively defined by

$$\overline{D}(x,E) := \limsup_{r\to 0^+} \frac{|E\cap B(x,r)|}{|B(x,r)|}, \quad \underline{D}(x,E) := \liminf_{r\to 0^+} \frac{|E\cap B(x,r)|}{|B(x,r)|}.$$

If the upper and lower densities are equal, their common value will be called the density of x at E and it will be denoted by $D(x,E)$. We shall use the word measurable to mean Lebesgue measurable.

Using densities we can define the essential interior $h_M(E)$, the essential closure $\overline{E}^M$ and the essential boundary $\partial^M E$ of a measurable set E as follows:

$$h_M(E) := \{x : D(x,E) = 1\}, \quad \overline{E}^M := \left\{x : \overline{D}(x,E) > 0\right\} \tag{18}$$

$$\partial^M E := \overline{E}^M \cap \overline{\mathbb{R}^N \setminus E}^M = \left\{x : \overline{D}(x,E) > 0,\, \overline{D}(x,\mathbb{R}^N\setminus E) > 0\right\}. \tag{19}$$

Notice also that by the Lebesgue differentiation theorem the symmetric difference $h_M(E)\Delta E$ is Lebesgue negligible, hence the measure theoretic interior of $h_M(E)$ is $h_M(E)$ (in this sense $h_M(E)$ is essentially open), and also that

$$\partial^M E = \mathbb{R}^N \setminus \big(h_M(E) \cup h_M(\mathbb{R}^N\setminus E)\big).$$

Here and in what follows we shall denote by $\mathcal{H}^\alpha$ the Hausdorff measure of dimension α in $\mathbb{R}^N$. In particular, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $\mathcal{H}^N$, the N-dimensional Hausdorff measure, coincides with the (outer) Lebesgue measure in $\mathbb{R}^N$. Given E_1, $E_2 \subseteq \mathbb{R}^N$, we shall write $E_1 = E_2 \pmod{\mathcal{H}^\alpha}$ if $H^\alpha(E_1\Delta E_2) = 0$, where $E_1\Delta E_2 = (E_1\setminus E_2)\cup(E_2\setminus E_1)$ is the symmetric difference of E_1 and E_2. We will use an analogous notation for the inclusion and in some cases, in order to simplify the notation, the equivalence or inclusion $\operatorname{mod}\mathcal{H}^N$ will be tacitly understood.

We say that a measurable set $E \subseteq \mathbb{R}^N$ has *finite perimeter* in $\mathbb{R}^N$ if there exist a positive finite measure μ in $\mathbb{R}^N$ and a Borel function $\nu_E\colon \mathbb{R}^N \to \mathbf{S}^{N-1}$ (called generalized inner normal to E) such that the following generalized Gauss-Green formula holds

$$\int_E \operatorname{div}\phi\, dx = -\int_{\mathbb{R}^N} \langle \nu_E, \phi\rangle\, d\mu \quad \forall\, \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N).$$

Hence the measure $\nu_E \mu$ is the distributional derivative of χ_E, which will be denoted by $D\chi_E$, while $\mu = |D\chi_E|$ is its total variation; the *perimeter* $P(E,B)$ of E in a Borel set $B \subseteq \mathbb{R}^N$ is defined by $|D\chi_E|(B)$, and we use the notation $P(E)$ in the case $B = \mathbb{R}^N$. For further information on sets of finite perimeter we refer to [**3**], [**13**], [**15**], [**17**], [**48**].

Definition 6. Let Ω be an open subset of $\mathbb{R}^N$. We say that a Borel function $u\colon \Omega \to [-\infty, +\infty]$ has weakly bounded variation in Ω if

$$P(\{u > t\}, \Omega) < \infty \quad \text{for a.e. } t \in \mathbb{R}.$$

The space of such functions will be denoted by $WBV(\Omega)$. We call total variation of u and denote by $|Du|$ the measure defined on every Borel subset $B \subset \Omega$ as

$$|Du|(B) := \int_{-\infty}^{+\infty} P(\{u > t\}, B)\, dt.$$

Remark 7. i) It follows from the properties of the perimeter that $|Du|$ is a σ-additive measure on the Borel sets of Ω. Let $BV(\Omega)$ denote the space of functions of bounded variation in Ω (see for instance [**3**], [**13**], [**15**], [**17**], [**48**]). Remark that by Lemma 1 in [**2**], $BV(\Omega) \subset WBV(\Omega)$. Furthermore, if Ω is bounded with Lipschitz boundary, $u \in WBV(\Omega)$ and $|Du|(\Omega) < +\infty$ then, by [**2**], $u \in BV(\Omega)$ and, by the coarea formula, $|Du|$ coincides with the total variation of u.

ii) It must be emphasized that WBV is *not* a vector space. Take indeed the two functions $u(x) = 1/x$ and $v(x) = 1/x - \sin(1/x)$ defined on $(-1,1)$. Then, clearly, $u, v \in WBV(-1,1)$ whereas $u - v \notin WBV(-1,1)$ since $\sin(1/x)$ assumes infinitely many times any value $t \in [-1,1]$. However, a strong motivation for the introduction of $WBV(\Omega)$ is the following result, showing that $WBV(\Omega)$ is the smallest space containing $BV(\Omega)$ and invariant under *any* contrast change; note that by Volpert's chain rule for distributional derivatives, $BV(\Omega)$ is stable only under *Lipschitz* contrast changes.

Theorem 3 ([**2**])**.** *For any $u \in WBV(\Omega)$ there exists a bounded, continuous and strictly increasing function $\phi\colon [-\infty, +\infty] \to \mathbb{R}$ such that $\phi \circ u \in BV(\Omega)$.*

5.2. *M*-connected components of sets of finite perimeter.

Let $E \subseteq \mathbb{R}^N$ be a set with finite perimeter. We say that E is *decomposable* if there exists a partition (A, B) of E such that $P(E) = P(A) + P(B)$ and both $|A|$ and $|B|$ are strictly positive. We say that E is *indecomposable* if it is not decomposable; notice that the properties of being decomposable or indecomposable are invariant $\mathrm{mod}\,\mathcal{H}^N$ and that, according to our definition, any Lebesgue negligible set is indecomposable.

In this section we want to recall the following decomposition theorem; a similar decomposition result for integer currents is stated in 4.2.25 of [**15**]. This result has also been used in G. Dolzmann and S. Müller [**11**] and B. Kirchheim [**20**] to prove Liouville type theorems for partial differential inclusions with multiple wells; the second paper contains also an explicit proof of the decomposition theorem, based on Lyapunov convexity theorem. The proof given in [**2**] was based on a simple variational argument.

Theorem 4 (Decomposition theorem)**.** *Let E be a set with finite perimeter in $\mathbb{R}^N$. Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i\in I}$ such that $|E_i| > 0$ and $P(E) = \sum_i P(E_i)$. Such that $\partial^M E = \cup_i \partial^M E_i$ $(\mathrm{mod}\,\mathcal{H}^{N-1})$,*

$$\mathcal{H}^{N-1}\left(h_M(E) \setminus \bigcup_{i\in I} h_M(E_i)\right) = 0 \tag{20}$$

and the E_i's are maximal indecomposable sets, i.e. any indecomposable set $F \subseteq E$ is contained $\mathrm{mod}\,\mathcal{H}^N$ *in some set E_i.*

Definition 7 (M-connected components)**.** In view of the previous theorem, we call the sets E_i the *M-connected components* of E and denote this family by $\mathcal{CC}^M(E)$; we always choose the index set I as an interval of $\mathbb{N}$, with $0 \in I$.

Notice that $\mathcal{CC}^M(E) = \emptyset$ whenever E is Lebesgue negligible and that Theorem 4 gives

$$\partial^M F \subset \partial^{\mathrm{M}} E \quad \mathrm{mod}\,\mathcal{H}^{N-1} \quad \text{for any } F \in \mathcal{CC}^M(E). \tag{21}$$

By (20), for $\mathcal{H}^{N-1}$-a.e. $x \in h_M(E)$ it also makes sense to talk about the *M-connected component of E containing* x, namely the unique set $F \in \mathcal{CC}^M(E)$ such that $x \in h_M(F)$.

Definition 8 (Holes, saturation)**.** Let E be an indecomposable set. We call *M-hole* of E any M-connected component of $\mathbb{R}^N \setminus E$ with finite measure. We define the saturation of E, denoted by $\mathrm{Sat}(E)$, as the union of E and of its M-holes. In the general case when E has finite perimeter, we define

$$\mathrm{Sat}(E) := \bigcup_{i\in I} \mathrm{Sat}(E_i) \quad \text{where} \quad \mathcal{CC}^M(E) = \{E_i\}_{i\in I}.$$

The M-holes of E, denoted by $H^M(E)$, is the family of M-holes of its M-connected components E_i.

Proposition 15. *Let $E, F \subseteq \mathbb{R}^N$ be two indecomposable sets. If $|E \cap F| = 0$, then the sets* $\mathrm{Sat}(E)$, $\mathrm{Sat}(F)$ *are either one a subset of the other, or they are disjoint* ($\mathrm{mod}\,\mathcal{H}^N$).

This was also proved in [**2**, Proposition 6].

Definition 9 (Exterior)**.** If $E \subseteq \mathbb{R}^N$ has finite perimeter and $|E| < \infty$, we call *exterior* of E the unique ($\mathrm{mod}\,\mathcal{H}^N$) M-component of $\mathbb{R}^N \setminus E$ with infinite measure. The exterior of E will be denoted by $\mathrm{ext}(E)$.

Notice that the notion of exterior makes sense only if $|E| < \infty$, due to the fact that $\mathbb{R}^N \setminus E$ has finite measure if $P(E) < \infty$ and $|E| = \infty$.

Definition 10 (Jordan boundary)**.** Any indecomposable subset of $\mathbb{R}^N$ such that $\mathrm{Sat}(E) = E$ will be called simple. We say that a set J is a Jordan boundary if there is a simple set E such that $J = \partial^{\mathrm{M}} E$ ($\mathrm{mod}\,\mathcal{H}^{N-1}$).

The simple set E associated to a Jordan boundary J is unique ([**2**, Proposition 7]). In this sense, J can also be thought as an *oriented* set, with the orientation induced by the generalized inner normal to E. The terminology was motivated by the results concerning sets in the plane, see in particular [**2**, Theorem 7]. We shall write $\mathrm{int}(J) = E$ and $\mathrm{ext}(J) = \mathbb{R}^N \setminus E$; notice that $\mathrm{ext}(J) = \mathrm{ext}(E)$.

Proposition 16. *Let E be indecomposable and let $\{Y_i\}_{i \in I}$ be its holes. Then*

$$E = \mathrm{Sat}(E) \setminus \bigcup_{i \in I} Y_i = \mathrm{Sat}(E) \cap \bigcap_{i \in I} \mathrm{ext}(Y_i) \tag{22}$$

and

$$P(E) = P(\mathrm{Sat}(E)) + \sum_{i \in I} P(Y_i). \tag{23}$$

There is also a converse statement of last result which can be seen in [**2**].

In order to simplify the following statement we enlarge the class of Jordan boundaries by introducing a *formal* Jordan boundary J_∞ whose interior is $\mathbb{R}^N$ and a *formal* Jordan boundary J_0 whose interior is empty; we also set $\mathcal{H}^{N-1}(J_\infty) = \mathcal{H}^{N-1}(J_0) = 0$ and denote by $\mathcal{S}$ this extended class of Jordan boundaries. In this way we are able to consider at the same time sets with finite and infinite measure and we can always assume that the list of components (or holes of the components) is infinite, possibly adding to it infinitely many $\mathrm{int}(J_0)$.

The following theorem describes $\partial^{\mathrm{M}} E$ by a collection of "external Jordan boundaries" J_i^+ and "internal Jordan boundaries" J_i^- satisfying some inclusion properties; these properties provide an axiomatic characterization of them [**2**]. However, we emphasize that in general this description is not invariant under complementation, i.e. the external (internal) boundaries of a set are not the internal (external) boundaries of the complement [**2**]. We shall see below that for almost all level sets λ of a function $u \in C(\overline{\Omega}) \cap WBV(\Omega)$, the external (internal) boundaries of $[u \geq \lambda]$ are the internal (external) boundaries of the complement $[u < \lambda]$.

Theorem 5 (Decomposition of $\partial^{\mathrm{M}} E$ in Jordan boundaries)**.** *Let $E \subseteq \mathbb{R}^{\mathrm{N}}$ be a set of finite perimeter. Then, there is a unique decomposition of $\partial^{\mathrm{M}} E$ into Jordan boundaries $\left\{J_i^+, J_k^- : i, k \in \mathbb{N}\right\} \subseteq \mathcal{S}$, such that*

(i) *Given* $\mathrm{int}(J_i^+)$, $\mathrm{int}(J_k^+)$, $i \neq k$, *they are either disjoint or one is contained in the other; given* $\mathrm{int}(J_i^-)$, $\mathrm{int}(J_k^-)$, $i \neq k$, *they are either disjoint or one is contained in the other. Each* $\mathrm{int}(J_i^-)$ *is contained in one of the* $\mathrm{int}(J_k^+)$.
(ii) $P(E) = \sum_i \mathcal{H}^{N-1}(J_i^+) + \sum_k \mathcal{H}^{N-1}(J_k^-)$.
(iii) *If* $\mathrm{int}(J_i^+) \subseteq \mathrm{int}(J_j^+)$, $i \neq j$, *then there is Jordan boundary* J_k^- *such that* $\mathrm{int}(J_i^+) \subseteq \mathrm{int}(J_k^-) \subseteq \mathrm{int}(J_j^+)$. *Similarly, if* $\mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_j^-)$, $i \neq j$, *then there is some Jordan boundary* J_k^+ *such that* $\mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_k^+) \subseteq \mathrm{int}(J_j^-)$.
(iv) *Setting* $L_j = \{i : \mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_j^+)\}$, *the sets* $Y_j = \mathrm{int}(J_j^+) \setminus \cup_{i \in L_j} \mathrm{int}(J_i^-)$ *are pairwise disjoint, indecomposable and* $E = \cup_j Y_j$.

Definition 11. Let E be a set of finite perimeter in $\mathbb{R}^N$. The family of boundaries J_i^+ described in last theorem will be called the family of external boundaries of E and will be denoted by $\mathcal{E}^M(E)$. The family of curves J_i^- will be called the internal boundaries of E and will be denoted by $\mathcal{I}^M(E)$.

Thus the external boundaries of E are the external boundaries of its M-connected components. The internal boundaries of E are the boundaries of the M-holes of its M-connected components. If E is indecomposable, then the external boundary of E is unique and coincides with $\partial^M \,\mathrm{Sat}(E)$.

It was shown in [**2**] that the boundary of a simple set in the plane was a rectifiable Jordan curve. Moreover, the M-components in the plane were also classically connected by arcs (indeed, a suitable representative).

5.3. M-components versus classical components.

Let $\overline{\Omega}$ be a Jordan domain in $\mathbb{R}^N$, i.e., the closure of the bounded connected component of the complement of a subset of $\mathbb{R}^N$ homeomorphic to S^{N-1}, the sphere of $\mathbb{R}^N$. Let Ω be the interior of $\overline{\Omega}$. We assume that the boundary of Ω is Lipschitz. Let $A \subseteq \Omega$. We write $A \subset\subset \Omega$ if there is some $\epsilon > 0$ such that $d(x, \mathbb{R}^N \setminus \Omega) \geq \epsilon$ for almost every $x \in A$. Let $E \subseteq \Omega$ be a set of finite perimeter in Ω such that

$$E \subset\subset \Omega \quad \text{or} \quad \Omega \setminus E \subset\subset \Omega. \tag{24}$$

Note that in this case, the M-holes of E are well defined as its M-holes as a subset of $\mathbb{R}^N$. Hence, also $\mathrm{Sat}(E)$ is well defined. If a set of finite perimeter E is such that $\Omega \setminus E \subset\subset \Omega$ then $\partial\Omega \subseteq \partial^M E$ and we shall exclude it as a curve of $\mathcal{E}^M(E)$.

We shall say that two sets A, B contained in a set $C \subseteq \mathbb{R}^2$ are *classically connected* inside C if there is a connected set $K \subseteq C$ containing A and B.

Definition 12. Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. We say that $\lambda \in R$ is a critical level of u if there exist $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$, $Y \neq Y'$, which are classically connected inside $[u \geq \lambda]$.

Theorem 6. *Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant on a neighborhood of $\partial\Omega$. Then the set of critical levels is of measure zero.*

Theorem 7. *Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant in a neighborhood of $\partial\Omega$. Then there is subset Λ of $\mathbb{R}$ such that $|\mathbb{R} \setminus \Lambda| = 0$, and, for every $\lambda \in \Lambda$, $\mathcal{E}^M([u \geq \lambda]) = \mathcal{I}^M([u < \lambda])$ and $\mathcal{I}^M([u \geq \lambda]) = \mathcal{E}^M([u < \lambda])$.*

Lemma 12. *Let X be a bounded indecomposable set of finite perimeter in $\mathbb{R}^N$ with all points of density 1. Suppose that $|\partial X| = 0$. Then $\overline{X}$ is connected.*

Proof: Suppose that $\overline{X}$ is not connected. Let X_1 and X_2 be two components of $\overline{X}$. Then there are two closed, disjoint sets H_1, H_2, whose union is $\overline{X}$ such that $X_1 \subseteq H_1$ and $X_2 \subseteq H_2$ ([**33**, Theorem 5.5, p. 82]). Since H_1 and H_2 are compact and disjoint they are at a positive distance from each other. Now, observe that both sets X_1 and X_2 contain points of X, hence points of density 1 in X, therefore, $|H_1| > 0$, $|H_2| > 0$. Since $|\partial X| = 0$, $\overline{X}$ is also a set of finite perimeter with $\partial^M \overline{X} = \partial^M X$. It follows that H_1 and H_2 are also sets of finite perimeter and $\mathrm{Per}(X) = \mathrm{Per}(H_1) + \mathrm{Per}(H_2)$. Hence, (H_1, H_2) is a decomposition of X. This contradiction proves that $\overline{X}$ must be connected. □

Lemma 13. *Let* $u \in WBV(\Omega) \cap C(\overline{\Omega})$. *Then* $IS_\epsilon u, SI^\epsilon u \in WBV(\Omega) \cap C(\overline{\Omega})$.

Proof: For any set $X \subseteq \mathbb{R}^N$, let $\mathrm{CC}(X)$ be the (classical) connected components of X. We already know that $IS_\epsilon u, SI^\epsilon u \in C(\overline{\Omega})$. Let $\lambda \in \mathbb{R}$ be such that $[u \leq \lambda]$ (resp. $[u \geq \lambda]$) is a set of finite perimeter such that $|[u = \lambda]| = 0$, and let $X \in \mathrm{CC}\ ([u \leq \lambda])$ (resp. $X \in \mathrm{CC}\ ([u \geq \lambda])$). Since, by Lemma 12, $\overline{Y}$ is connected if $Y \in \mathcal{CC}^M([u \leq \lambda])$ (resp. $Y \in \mathcal{CC}^M$ $([u \geq \lambda])$) is such that all its points are of density 1, then we have that either $Y \subseteq X$ or $X \cap Y = \emptyset \ (\mathrm{mod}\, \mathcal{H}^N)$. From this it follows that each set $X \in \mathrm{CC}\ ([u \leq \lambda])$ (resp. $X \in \mathrm{CC}\ ([u \geq \lambda])$) is a set of finite perimeter which is a union of M-components of $[u \leq \lambda]$ (resp. $[u \geq \lambda]$). Moreover

$$P([u \leq \lambda], \Omega) = \sum_{X \in \mathrm{CC}([u \leq \lambda]),\, |X| > 0} P(X, \Omega) \tag{25}$$

(resp.

$$P([u \geq \lambda], \Omega) = \sum_{X \in \mathrm{CC}([u \geq \lambda]),\, |X| > 0} P(X, \Omega)\ ). \tag{26}$$

Now, recall that by Lemma 1 we have that $(\mathrm{mod}\, \mathcal{H}^N)$

$$[IS_\epsilon u < \lambda] \subseteq T'_\epsilon [u \leq \lambda] \subseteq [IS_\epsilon u \leq \lambda].$$

If $\lambda \in \mathbb{R}$ is such that $|[IS_\epsilon u = \lambda]| = 0$, then

$$[IS_\epsilon u \leq \lambda] = T'_\epsilon [u \leq \lambda] = \cup_{X \in \mathrm{CC}([u \leq \lambda]),\, |X| > \epsilon} X \quad (\mathrm{mod}\, \mathcal{H}^N).$$

Since $u \in WBV(\Omega)$ and $[u \leq \lambda]$ is a set of finite perimeter, we conclude from last equality and (25) that $[IS_\epsilon u \leq \lambda]$ is a set of finite perimeter. Hence, the sets $[IS_\epsilon u \leq \lambda]$ are sets of finite perimeter for almost all levels $\lambda \in \mathbb{R}$. Similarly, we prove that $SI^\epsilon u \in WBV(\Omega)$. □

Let A be an indecomposable set of finite perimeter in $\mathbb{R}^N$. The set $\mathrm{Sat}(A)$ is defined modulo a null set. Let us define a representative of it in case that all points of A are of density 1, and $|\partial A| = 0$. Recall that $\mathrm{Sat}(A)$ is equal to A union the M-components of $\mathbb{R}^N \setminus A$ of finite measure. We define a set $\mathrm{Sat}^*(A)$ which coincides with $\mathrm{Sat}(A)$ modulo an $\mathcal{H}^N$ null set. For that, we take $\overline{A}$ as a representative of A, since $|\partial A| = 0$, and observe that $\mathbb{R}^N \setminus \overline{A}$ is an open set. Since each connected component of $\mathbb{R}^N \setminus \overline{A}$ is either contained or disjoint to a given M-component of $\mathbb{R}^N \setminus \overline{A}$, then each M-component of $\mathbb{R}^N \setminus \overline{A}$ is the union of a, at most countable, family of connected components of $\mathbb{R}^N \setminus \overline{A}$. Let B_n be the M-components of $\mathbb{R}^N \setminus \overline{A}$ of finite measure. For each B_n there is a family of components B_n^m of $\mathbb{R}^N \setminus \overline{A}$ such that $B_n = \cup_m B_n^m$

$(\operatorname{mod} \mathcal{H}^N)$. Then, we define $\operatorname{Sat}^*(A) = A \cup \cup_{n,m} B_n^m$. Since $\partial B_{n,m} \subseteq \partial A$, we have $\partial(\cup_{n,m} B_n^m) \subseteq \overline{\cup_{n,m} \partial B_{n,m}} \subseteq \partial A$ ([**22**, vol. II, p. 168]). Hence $\partial \operatorname{Sat}^*(A) \subseteq \partial A$. Thus $|\partial \operatorname{Sat}^*(A)| = 0$. The assumptions of Lemma 12 are satisfied, therefore, $\overline{\operatorname{Sat}^*(A)}$ is connected. Similarly, let B_∞ be the M-component of $\mathbb{R}^N \setminus \overline{A}$ of infinite measure and let B_∞^m be the components of $\mathbb{R}^N \setminus \overline{A}$ whose union coincides with B_∞ modulo a null set. Then we define $\operatorname{SAT}(A)$ by the identity $\mathbb{R}^N \setminus \operatorname{SAT}(A) = \cup_m B_\infty^m$. Observe that $\operatorname{SAT}(A) \subseteq \operatorname{sat}(\overline{A})$.

Lemma 14. i) *Let A be an indecomposable set of finite perimeter such that all its points are of density* 1 *and* $|\partial A| = 0$. *Then* $\operatorname{SAT}(A) = \overline{\operatorname{Sat}^*(A)}$. *Thus,* $\operatorname{SAT}(A)$ *is connected.*

ii) *The set* $\overline{\cup_m \partial B_\infty^m}$ *is connected.*

iii) *We have that* $\partial \operatorname{SAT}(A) = \overline{\cup_m \partial B_\infty^m}$.

Proof: i) By definition, we have that $\operatorname{Sat}^*(A) \subseteq \operatorname{SAT}(A)$. Since $\operatorname{SAT}(A)$ is closed, we have that $\overline{\operatorname{Sat}^*(A)} \subseteq \operatorname{SAT}(A)$. To check the opposite inclusion, let $p \notin \overline{\operatorname{Sat}^*(A)}$. Since $\overline{A} \subseteq \overline{\operatorname{Sat}^*(A)}$, there is some $r > 0$ such that $B(p,r) \cap \overline{A} = \emptyset$, $B(p,r) \cap (\cup_{n,m} B_n^m) = \emptyset$. Then $B(p,r) \subseteq \mathbb{R}^N \setminus (\overline{A} \cup \cup_{n,m} B_n^m) = \cup_m B_\infty^m = \mathbb{R}^N \setminus \operatorname{SAT}(A)$. Thus, $B(p,r) \cap \operatorname{SAT}(A) = \emptyset$, and, in particular, $p \notin \operatorname{SAT}(A)$. We have proved that $\operatorname{SAT}(A) = \overline{\operatorname{Sat}^*(A)}$.

ii) Suppose that $\overline{\cup_m \partial B_\infty^m}$ is not connected. Since this set is compact, there exist two nonempty closed disjoint sets H_1, H_2 whose union is $\overline{\cup_m \partial B_\infty^m}$. Let $I_1 := \{m : \partial B_\infty^m \cap H_1 \neq \emptyset\}$, $I_2 := \{m : \partial B_\infty^m \cap H_2 \neq \emptyset\}$. Since the sets ∂B_∞^m are connected, then $I_1 \cap I_2 = \emptyset$. We have

$$\overline{\cup_{m \in I_1} \partial B_\infty^m} \subseteq H_1, \quad \overline{\cup_{m \in I_2} \partial B_\infty^m} \subseteq H_2.$$

Since

$$\overline{\cup_{m \in \mathbb{N}} \partial B_\infty^m} \subseteq \overline{\cup_{m \in I_1} \partial B_\infty^m} \cup \overline{\cup_{m \in I_2} \partial B_\infty^m},$$

if $p \in H_1$ and $p \notin \overline{\cup_{m \in I_1} \partial B_\infty^m}$, then $p \in \overline{\cup_{m \in I_2} \partial B_\infty^m} \subseteq H_2$. Thus $H_1 \cap H_2 \neq \emptyset$. This contradiction proves that

$$H_1 = \overline{\cup_{m \in I_1} \partial B_\infty^m}, \quad H_2 = \overline{\cup_{m \in I_2} \partial B_\infty^m}.$$

Since $d(H_1, H_2) > 0$, the sets $U_1 = \cup_{m \in I_1} B_\infty^m$ and $U_2 = \cup_{m \in I_2} B_\infty^m$ constitute a decomposition of $\mathbb{R}^N \setminus \operatorname{SAT}(A)$, an indecomposable set. This contradiction proves the first assertion of the lemma.

iii) From the identity $\mathbb{R}^N \setminus \mathrm{SAT}(A) = \cup_m B^m_\infty$, we deduce that $\partial\,\mathrm{SAT}(A) \subseteq \overline{\cup_m \partial B^m_\infty}$. The other inclusion is a consequence of the following simple fact: if $X, O \subseteq \mathbb{R}^N$, O is open and $X \cap O = \emptyset$, then also $\overline{X} \cap O = \emptyset$. Indeed, since $B^m_\infty \cap \mathrm{SAT}(A) = \emptyset$, then also $B^m_\infty \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$. Hence, $\overline{B^m_\infty} \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$. In particular, we have that $\partial B^m_\infty \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$. Since this is true for all m, we have that $\cup_m \partial B^m_\infty \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$. Again, by the simple fact mentioned above, we have $\overline{\cup_m \partial B^m_\infty} \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$. Now, since $\partial B^m_\infty \subseteq \partial A$ for all m, we have that $\overline{\cup_m \partial B^m_\infty} \subseteq \partial A \subseteq \mathrm{SAT}(A)$. From both facts, we deduce that $\overline{\cup_m \partial B^m_\infty} \subseteq \partial\,\mathrm{SAT}(A)$. □

Lemma 15. *Let A be an indecomposable set of finite perimeter such that all its points are of density 1 and $|\partial A| = 0$. Let K be a continuum. Assume that $K \cap \mathrm{SAT}(A) \neq \emptyset$. Then $(K \setminus \mathrm{int}(\mathrm{SAT}(A))) \cup \partial\,\mathrm{SAT}(A)$ is connected.*

Proof: Since $K \cap \mathrm{SAT}(A) \neq \emptyset$, then $K \cup \mathrm{SAT}(A)$ is connected. If $K \setminus \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$, then $(K \setminus \mathrm{int}(\mathrm{SAT}(A))) \cup \partial\,\mathrm{SAT}(A) = \partial\,\mathrm{SAT}(A)$, which, by last lemma, is a connected set. Suppose that $K \setminus \mathrm{int}(\mathrm{SAT}(A)) \neq \emptyset$ and the set $(K \setminus \mathrm{int}(\mathrm{SAT}(A))) \cup \partial\,\mathrm{SAT}(A)$ is not connected. Let K_1, K_2 be two connected components of $(K \setminus \mathrm{int}(\mathrm{SAT}(A))) \cup \partial\,\mathrm{SAT}(A)$ and let $a \in K_1$, $b \in K_2$. Then there are two closed, disjoint sets H_1, H_2, whose union is $(K \setminus \mathrm{int}(\mathrm{SAT}(A))) \cup \partial\,\mathrm{SAT}(A)$ such that $a \in H_1$ and $b \in H_2$ ([**33**, Theorem 5.5, p. 82]). Then, since K_1, K_2 are connected, then $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$. Without loss of generality, we may assume that $H_1 \cap \partial\,\mathrm{SAT}(A) \neq \emptyset$. Since $\partial\,\mathrm{SAT}(A)$ is connected, then $\partial\,\mathrm{SAT}(A) \subseteq H_1$, and, therefore, $H_2 \cap \partial\,\mathrm{SAT}(A) = \emptyset$. This implies that the set $H_2 \setminus \mathrm{SAT}(A)$ is closed. Then $a \in H_1 \cup \mathrm{SAT}(A)$, $b \in H_2 \setminus \mathrm{SAT}(A)$ are two nonempty and disjoint closed subsets of $K \cup \mathrm{SAT}(A)$, a contradiction with the conectedness of $K \cup \mathrm{SAT}(A)$. This contradiction proves the lemma. □

Lemma 16. *Let A, B be two indecomposable subsets of $\mathbb{R}^N$ of finite perimeter. Assume that all points of A, B are of density 1 and $|\partial A| = |\partial B| = 0$. Assume that $\overline{A} \cap \overline{B} = \emptyset$. Then*

(i) *if* $\mathrm{Sat}(A) \subseteq \mathrm{Sat}(B) \pmod{\mathcal{H}^N}$, *then* $\mathrm{SAT}(A) \subseteq \mathrm{int}(\mathrm{SAT}(B))$,
(ii) *if* $\mathrm{Sat}(B) \subseteq \mathrm{Sat}(A) \pmod{\mathcal{H}^N}$, *then* $\mathrm{SAT}(B) \subseteq \mathrm{int}(\mathrm{SAT}(A))$,
(iii) *if* $\mathrm{Sat}(A) \cap \mathrm{Sat}(B) = \emptyset \pmod{\mathcal{H}^N}$, *then* $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) = \emptyset$.

Proof: (i) Let $p \in A$. Since $D(p, A) = 1$, then also $D(p, \mathrm{SAT}(B)) = 1$. We have that $p \in \overline{\mathrm{SAT}(B)} = \mathrm{SAT}(B)$. Hence $A \subseteq \mathrm{SAT}(B)$, and, therefore, $\overline{A} \subseteq \mathrm{SAT}(B)$. Let B^m_∞ be the connected components of $\mathbb{R}^N \setminus \overline{B}$ whose union is $\mathbb{R}^N \setminus \mathrm{SAT}(B)$. Since $\mathbb{R}^N \setminus \mathrm{SAT}(B) \subseteq \mathbb{R}^N \setminus \overline{A}$ and B^m_∞ is open and connected, there is a connected component A_∞^m of $\mathbb{R}^N \setminus \overline{A}$ such that $B_\infty^m \subseteq A_\infty^m$. Since $\cup_m B_\infty^m \subseteq \cup_m A_\infty^m$ and $\cup_m B_\infty^m$ is M-connected, then it is easy to see that $\cup_m A_\infty^m$ is also M-connected (and has infinite measure). Thus

$$\mathbb{R}^N \setminus \mathrm{SAT}(B) = \cup_m B_\infty^m \subseteq \cup_m A_\infty^m \subseteq \mathbb{R}^N \setminus \mathrm{SAT}(A).$$

We obtain that $\mathrm{SAT}(A) \subseteq \mathrm{SAT}(B)$. Since $\partial\,\mathrm{SAT}(A) \subseteq \partial A$, $\partial\,\mathrm{SAT}(B) \subseteq \partial B$ and $\overline{A} \cap \overline{B} = \emptyset$, we obtain that $\mathrm{SAT}(A) \subseteq \mathrm{int}(\mathrm{SAT}(B)$.

Since the case (ii) follows from case (i) by symmetry, we consider case (iii). Since $\overline{A} \cap \overline{B} = \emptyset$, then $\overline{B}$ is contained in a connected component, say C, of $\mathbb{R}^N \setminus \overline{A}$, which is an open set. Then C is open and M-connected, and, thus, contained in the union of open and connected components of $\mathbb{R}^N \setminus \overline{A}$ that constitute an M-component D of $\mathbb{R}^N \setminus \overline{A}$. If $D = \mathbb{R}^N \setminus \mathrm{SAT}(A)$, then $\overline{B} \cap \mathrm{SAT}(A) = \emptyset$. If $D \subseteq \mathrm{SAT}(A)$, then $\overline{B} \subseteq \mathrm{SAT}(A)$, a contradiction with (iii). Thus, we conclude that

$$\overline{B} \cap \mathrm{SAT}(A) = \emptyset. \tag{27}$$

By symmetry, also

$$\overline{A} \cap \mathrm{SAT}(B). \tag{28}$$

Now, since $\mathrm{SAT}(A) = \overline{A} \cup A_0$ and $\mathrm{SAT}(B) = \overline{B} \cup B_0$, where A_0, B_0 are open sets, using (27) and (28), we deduce that

$$\mathrm{SAT}(A) \cap \mathrm{SAT}(B) = A_0 \cap B_0.$$

Thus, if $\mathrm{SAT}(A)$ and $\mathrm{SAT}(B)$ intersect, they do it in an open set, which is of positive measure. This contradicts our assumption that $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) = \emptyset \pmod{\mathcal{H}^N}$. Therefore, $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) = \emptyset$. □

Lemma 17. *Let A, B be two indecomposable subsets of $\mathbb{R}^N$ of finite perimeter. Assume that all points of A, B are of density 1 and $|\partial A| = |\partial B| = 0$. Suppose that $\mathrm{Sat}(A) \cap \mathrm{Sat}(B) = \emptyset \pmod{\mathcal{H}^N}$ and $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) \neq \emptyset$. Then $\partial\,\mathrm{SAT}(A) \cap \partial\,\mathrm{SAT}(B) \neq \emptyset$.*

Under the assumptions of the lemma, the condition $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) \neq \emptyset$ is equivalent to $\overline{A} \cap \overline{B} \neq \emptyset$.

Proof: Since $\mathrm{Sat}(A) \cap \mathrm{Sat}(B) = \emptyset \pmod{\mathcal{H}^N}$ we have that $\mathrm{int}(\mathrm{SAT}(A)) \cap \mathrm{int}(\mathrm{SAT}(B)) = \emptyset$. Now, suppose that $\partial\,\mathrm{SAT}(A) \cap \mathrm{int}(\mathrm{SAT}(B)) \neq \emptyset$. We observe that if $X = \overline{Z}$ then $\partial X \subseteq \partial Z$. Since $\mathrm{SAT}(A) = \overline{\mathrm{Sat}^*(A)}$, if $p \in \partial\,\mathrm{SAT}(A) \cap \mathrm{int}(\mathrm{SAT}(B))$, then $p \in \partial\,\mathrm{Sat}^*(A) \cap \mathrm{int}(\mathrm{SAT}(B))$. Let $r > 0$ be such that $B(p,r) \subseteq \mathrm{int}(\mathrm{SAT}(B))$. Then $B(p,r) \cap \mathrm{Sat}^*(A) \neq \emptyset$. Recall that $\mathrm{Sat}^*(A) = A \cup \cup_{n,m} A_n^m$ where A_n are the M-components $\mathbb{R}^N \setminus \overline{A}$ of finite measure and A_n^m are the (open) components de $\mathbb{R}^N \setminus \overline{A}$

such that $A_n = \cup_m A_n^m \pmod{\mathcal{H}^N}$. Thus either $B(p,r) \cap A \neq \emptyset$ or $B(p,r) \cap A_n^m \neq \emptyset$ for some n, m. Since all points of A are of density 1, if $B(p,r) \cap A \neq \emptyset$, then $|B(p,r) \cap A| > 0$. Hence, $|\operatorname{Sat}(A) \cap \operatorname{Sat}(B)| \geq |A \cap \operatorname{int}(\mathrm{SAT}(B))| > 0$, a contradiction. If $B(p,r) \cap A_n^m \neq \emptyset$, then $|\operatorname{Sat}(A) \cap \operatorname{Sat}(B)| \geq |A_n^m \cap B(p,r)| > 0$. This contradiction proves that $\partial\,\mathrm{SAT}(A) \cap \operatorname{int}(\mathrm{SAT}(B)) = \emptyset$. By symmetry, also $\operatorname{int}(\mathrm{SAT}(A)) \cap \partial\,\mathrm{SAT}(B) = \emptyset$. Therefore $\partial\,\mathrm{SAT}(A) \cap \partial\,\mathrm{SAT}(B) = \mathrm{SAT}(A) \cap \mathrm{SAT}(B) \neq \emptyset$. □

Let A, B be two indecomposable sets of finite perimeter in $\mathbb{R}^N$ such that all its points are of density 1, $|\partial A| = |\partial B| = 0$, and $A \cap B = \emptyset$. We have the three possibilities described in Lemma 16. If $\mathrm{SAT}(A) \cap \mathrm{SAT}(B) = \emptyset \pmod{\mathcal{H}^N}$, then we define $Q(A) = \partial\,\mathrm{SAT}(A)$, $Q(B) = \partial\,\mathrm{SAT}(B)$. If $\mathrm{SAT}(A) \subseteq \mathrm{SAT}(B) \pmod{\mathcal{H}^N}$, then $\mathrm{SAT}(A)$ is contained in a M-hole H of B. Take a point $p \in H \setminus \mathrm{SAT}(A)$ and, by an inversion φ_p with respect to p, we map A, B to some sets $\varphi_p(A)$, $\varphi_p(B)$ such that $\mathrm{SAT}(\varphi_p(A)) \cap \mathrm{SAT}(\varphi_p(B)) = \emptyset \pmod{\mathcal{H}^N}$. Then we define $\mathrm{SAT}(B,H) = \varphi_p^{-1}(\mathrm{SAT}(\varphi_p(B)))$, $\mathrm{SAT}(A,H) = \varphi_p^{-1}(\mathrm{SAT}(\varphi_p(A))) = \mathrm{SAT}(A)$, and $Q(A) = \partial\,\mathrm{SAT}(A)$, $Q(B) = \partial\,\mathrm{SAT}(A,H)$. In a similar way, we define $Q(A)$ and $Q(B)$ when $\mathrm{SAT}(B) \subseteq \mathrm{SAT}(A)$.

Lemma 18. *Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant on a neighborhood of $\partial\Omega$. Let $\lambda \in \mathbb{R}$ be such that $[u \geq \lambda]$ is a set of finite perimeter and $|[u = \lambda]| = 0$. Suppose that X is a connected component of $[u \geq \lambda]$ which is not M-connected. Then there exist $Y, Y' \in \mathcal{CC}^M(X)$, $Y \neq Y'$, which are classically connected inside X by a continuum $C \subseteq [u = \lambda]$ such that $C \supseteq Q(Y) \cup Q(Y')$.*

Proof: Using the same argument used in Lemma 13, under the assumptions of the present lemma, X is a set of finite perimeter in $\mathbb{R}^N$. Let Z be any M-component of X with $|Z| > 0$. We assume that all points of Z are of density 1 and, since $\partial Z \subseteq [u = \lambda]$, we have that $|\partial Z| = 0$. Observe that, if A, B are two such M-components, then $A \cap B = \emptyset$. Suppose that $\overline{A} \cap \overline{B} \neq \emptyset$. If $\operatorname{Sat}(A) \cap \operatorname{Sat}(B) = \emptyset \pmod{\mathcal{H}^N}$, then, by Lemma 17, we have that $\partial\,\mathrm{SAT}(A) \cap \partial\,\mathrm{SAT}(B) \neq \emptyset$ and we may take $Q(A) = \partial\,\mathrm{SAT}(A)$, $Q(B) = \partial\,\mathrm{SAT}(B)$, $C = Q(A) \cup Q(B)$. If $\operatorname{Sat}(A)$ is contained in an M-hole H of B, then, by an inversion with respect to a point $p \in H \setminus \mathrm{SAT}(A)$, we may reduce this case to the previous one. The case where $\operatorname{Sat}(B)$ is contained in an M-hole of A is similar to the previous one. Thus, we may assume that $\overline{A} \cap \overline{B} = \emptyset$, for any two M-components A, B of X.

Let F_0 be the M-component of $\mathbb{R}^N \setminus X$ containing the infinity. By the results of [**2**, Theorem 6], either i) there exist two M-components X_1, X_2 of X such that $\partial^M X_i \cap \partial^M F_0 \neq \emptyset$, $i = 1, 2$, or ii) there is only one of them X_1, in which case, there is an M-component F_1 of $\mathbb{R}^N \setminus X$ and an M-component X_2 of X, $X_2 \neq X_1$, such that $\partial^M X_1 \cap \partial^M F_1 \neq \emptyset$, $\partial^M X_2 \cap \partial^M F_1 \neq \emptyset$. Without loss of generality, we may assume that all points of X_1 and X_2 are of density 1. Moreover, since $\partial X_i \subseteq [u = \lambda]$ we have that $|\partial X_i| = 0$, $i = 1, 2$. Let Y, Y' be, respectively, the closures of X_1 and X_2. By Lemma 12, we know that Y, Y' are connected. By an inversion with respect to a point $p \in F_1$, $p \notin Y \cup Y'$, if necessary, we may assume that Y, Y' are always in the situation described in i). In particular, we have that

$$\mathrm{Sat}(Y) \cap \mathrm{Sat}(Y') = \emptyset \pmod{\mathcal{H}^N}. \tag{29}$$

Then, by Lemma 16, we know that $\mathrm{SAT}(Y) \cap \mathrm{SAT}(Y') = \emptyset$. Since X is connected, there exists a continuum $J \subseteq X$ such that $J \cap Y \neq \emptyset$, $J \cap Y' \neq \emptyset$. Let us define $J_1 = (J \setminus \mathrm{int}(\mathrm{SAT}(Y))) \cup \partial\,\mathrm{SAT}(Y)$, and $J_2 = (J_1 \setminus \mathrm{int}(\mathrm{SAT}(Y'))) \cup \partial\,\mathrm{SAT}(Y')$. Since $J \cap \mathrm{SAT}(Y) \supseteq J \cap Y \neq \emptyset$, J_1 is a continuum, by Lemma 15. Now, using (29), we have that $J_1 \cap \mathrm{SAT}(Y') \supseteq J \cap \mathrm{SAT}(Y') \supseteq J \cap Y' \neq \emptyset$ and, therefore, J_2 is a continuum, by Lemma 15. By construction, J_2 is disjoint to $\mathrm{int}(\mathrm{SAT}(Y))$ and $\mathrm{int}(\mathrm{SAT}(Y'))$. Let us write $K = J_2$.

Let A be an M-component of X with $|A| > 0$. Recall that we may assume that all its points are of density 1 and $|\partial A| = 0$. We are assuming that $\overline{A} \cap Y = \emptyset$, $\overline{A} \cap Y' = \emptyset$. Given such an A, by Lemma 16 applied to A, Y we have that either $\overline{A} \subseteq \mathrm{int}(\mathrm{SAT}(Y))$, or $\overline{Y} \subseteq \mathrm{int}(\mathrm{SAT}(A))$, or $\mathrm{SAT}(A) \cap \mathrm{SAT}(Y) = \emptyset$. Note that the second possibility cannot happen, since then $\partial^M Y$ would not intersect $\partial^M F_0$. Similar relations hold between A and Y'. Thus, the sets

$$\begin{aligned}
\mathcal{I}_0 = \{A : A \text{ is an } M \text{ component of } X,\ |A| > 0,&\\
\overline{A} \cap \mathrm{SAT}(Y) = \emptyset,\ \overline{A} \cap \mathrm{SAT}(Y') = \emptyset\},&\\
\mathcal{I}_1 = \{A : A \text{ is an } M \text{ component of } X,\ |A| > 0,\ \overline{A} \subseteq \mathrm{int}(\mathrm{SAT}(Y))\},&\\
\mathcal{I}_2 = \{A : A \text{ is an } M \text{ component of } X,\ |A| > 0,\ \overline{A} \subseteq \mathrm{int}(\mathrm{SAT}(Y'))\},&
\end{aligned}$$

contain all M-components of X (we assume always that all points of A are of density 1), other than Y, Y'.

We observe that, if $A \in \mathcal{I}_0$, by Lemma 14, the set $\partial\,\mathrm{SAT}(A)$ is a continuum, which is contained in $[u = \lambda]$. Let $\{A_n\}$ be the elements of $\mathcal{I}_0$ such that $\mathrm{SAT}(A_n) \cap K \neq \emptyset$. We define inductively a sequence K_n.

Let $K_0 = K$. Suppose that we have defined K_i for $i \leq n-1$. If

$$\mathrm{SAT}(A_n) \subseteq \mathrm{int}(\mathrm{SAT}(A_i)) \quad \text{for some } i < n, \tag{30}$$

we define $K_n = K_{n-1}$. Thus, by Lemma 16, we may assume that, given $i < n$, either $\mathrm{SAT}(A_i) \subseteq \mathrm{SAT}(A_n)$ or $\mathrm{SAT}(A_n) \cap \mathrm{SAT}(A_i) = \emptyset$. In this case, we define $K_n = (K_{n-1} \backslash \mathrm{int}(\mathrm{SAT}(A_n))) \cup \partial \, \mathrm{SAT}(A_n)$. In this way, we may discard all sets A_n satisfying (30). If $\mathrm{SAT}(A_{n-1}) \subseteq \mathrm{int}(\mathrm{SAT}(A_n))$, then $\mathrm{SAT}(A_n) \cap \partial \, \mathrm{SAT}(A_{n-1}) \neq \emptyset$, and, thus $\mathrm{SAT}(A_n) \cap K_{n-1} \neq \emptyset$. If $\mathrm{SAT}(A_n) \cap \mathrm{SAT}(A_{n-1}) = \emptyset$, then $\mathrm{SAT}(A_n) \cap K_{n-1} = \mathrm{SAT}(A_n) \cap K_{n-2} \cup \mathrm{SAT}(A_n) \cap \partial \, \mathrm{SAT}(A_{n-1}) \supseteq \mathrm{SAT}(A_n) \cap K_{n-2}$. Iteratively, we get that either $\mathrm{SAT}(A_n) \cap K_{n-i} \neq \emptyset$ or $\mathrm{SAT}(A_n) \cap K_{n-1} \supseteq \mathrm{SAT}(A_n) \cap K_{n-i-1}$. Since $\mathrm{SAT}(A_n) \cap K \neq \emptyset$, we conclude that $\mathrm{SAT}(A_n) \cap K_{n-1} \neq \emptyset$ for all $n \geq 1$. Now, by Lemma 15, we obtain that K_n are continua for all $n \geq 1$. Moreover, for all $n \geq 1$, K_n contains $\partial \, \mathrm{SAT}(Y)$ and $\partial \, \mathrm{SAT}(Y')$. According to Blaschke selection Theorem, K_n has a subsequence converging in the Hausdorff metric ([**14**, Theorem 3.16]). Let C be a limit of K_n. Then C is a continuum joining Y to Y' contained in X ([**22**, vol. II, p. 111], [**14**, Theorem 3.18]). Let $i \in \mathbb{N}$. Since $K_n \cap \mathrm{int}(\mathrm{SAT}(A_i)) = \emptyset$ for all $n \geq i$, and $K_n \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$ for all $A \in \mathcal{I}_0$ such that $\mathrm{SAT}(A) \cap K = \emptyset$, then $C \cap \mathrm{int}(\mathrm{SAT}(A_i)) = \emptyset$ and $C \cap \mathrm{int}(\mathrm{SAT}(A)) = \emptyset$ for all $A \in \mathcal{I}_0$ such that $\mathrm{SAT}(A) \cap K = \emptyset$. Also $C \cap \mathrm{int}(\mathrm{SAT}(Y)) = C \cap \mathrm{int}(\mathrm{SAT}(Y')) = \emptyset$. Since the set

$$X \cap [u > \lambda] \subseteq \cup_{A \in \mathcal{I}_0} \mathrm{int}(\mathrm{SAT}(A)) \cup \mathrm{int}(\mathrm{SAT}(Y)) \cup \mathrm{int}(\mathrm{SAT}(Y')),$$

we conclude that $C \subseteq [u = \lambda]$. □

Observe that if $u \in WBV(\Omega) \cap C(\overline{\Omega})$ is constant in a neighborhood of $\partial\Omega$ then $[u \geq \lambda]$ satisfies (24) for all $\lambda \in \mathbb{R}$. Let D be the set of $\lambda \in \mathbb{R}$ such that $[u \geq \lambda]$ is not a set of finite perimeter or $|[u = \lambda]| > 0$. Then $D \subset \mathbb{R}$ is a set of measure 0. Let $\lambda \notin D$. If $\lambda > \sup_{x \in \Omega} u(x)$, then $[u < \lambda] = \Omega$, $[u \geq \lambda] = \emptyset$, λ is not a critical level and the assertion of Theorem 7 holds. If $\lambda \leq \inf_{x \in \Omega} u(x)$, then $[u < \lambda] = \emptyset$, $[u \geq \lambda] = \Omega$ and again our assertions hold.

Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant in a neighborhood of $\partial\Omega$. Let $R(\partial\Omega)$ be the connected region of Ω containing $\partial\Omega$ where u is a constant. Let us observe that, if $X \in \mathcal{CC}^M([u \leq \lambda])$ or $X \in \mathcal{CC}^M([u \geq \lambda])$ and $J \in \mathcal{E}^M(X) \cup \mathcal{I}^M(X)$, then $J \subset\subset \Omega$. Indeed, either $R(\partial\Omega) \subseteq X$ or $R(\Omega) \cap X = \emptyset$. Since we have excluded $\partial\Omega$ as an exterior boundary, then in both cases we have that $J \subset\subset \Omega$. For simplicity of notation, let us mention that, when we write $\partial^M X$, for a set X like above, we exclude $\partial\Omega$ as being part of $\partial^M X$.

Lemma 19. *Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant in a neighborhood of $\partial\Omega$. Let $\lambda \in (\inf_\Omega u, \sup_\Omega u)$ be such that $[u \geq \lambda]$ is a set of finite perimeter and $|[u = \lambda]| = 0$. Let $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$, $Y \neq Y'$. We assume that Y and Y' are the closures of sets with all points of density 1 and $|\partial Y| = |\partial Y'| = 0$. Suppose that Y, Y' are classically connected inside $[u \geq \lambda]$ by a continuum $C \subseteq [u = \lambda]$ such that $C \supseteq Q(Y) \cup Q(Y')$. Then*

(i) *for $\epsilon \in \{1/n : n \geq 1\}$ small enough, $Y, Y' \in \mathcal{CC}^M([SI^\epsilon u \geq \lambda])$ and are classically connected inside $[SI^\epsilon u \geq \lambda]$. Moreover $SI^\epsilon u = \lambda$ on C.*

(ii) *Assume that $|[IS_\epsilon u = \lambda]| = 0$ for any $\epsilon \in \{1/n : n \geq 1\}$. Then for $\epsilon \in \{1/n : n \geq 1\}$ small enough, Y, Y' are not M-connected as subsets (i.e., are contained in different indecomposable subsets) of $[IS_\epsilon u \geq \lambda]$ but are classically connected there by C. Moreover $IS_\epsilon u = \lambda$ on C.*

Proof: (i) Let $\epsilon \in \{1/n : n \geq 1\}$, $\epsilon < |Y|, |Y'|$. We know that the sets Y, Y' can be considered as closed sets. Since Y and Y' are classically connected inside $[u \geq \lambda]$, then there is a connected component X of $[u \geq \lambda]$ containing them. Observe that $X \in \mathcal{F}^\epsilon(u, x)$, for all $x \in X$ and, therefore, $SI^\epsilon(u)(x) \geq \lambda$ for all $x \in X$. In particular, Y and Y' are classically connected inside $[SI^\epsilon u \geq \lambda]$.

Let us prove that $Y, Y' \in \mathcal{CC}^M([SI^\epsilon u \geq \lambda])$. For that we prove that $\partial^M Y, \partial^M Y' \subseteq \partial^M [SI^\epsilon u < \lambda] \ (\mathrm{mod}\, \mathcal{H}^{N-1})$. The argument in both cases being similar, we shall only consider the case for Y in detail. For $\mathcal{H}^{N-1}$-almost all $p \in \partial^M Y$ we have $\overline{D}(p, [u < \lambda]) = \overline{D}(p, [u \leq \lambda]) > 0$, $\overline{D}(p, Y) > 0$. Since $SI^\epsilon u \leq u$ we also have $\overline{D}(p, [SI^\epsilon u < \lambda]) > 0$ and, we have $\overline{D}(p, \mathbb{R}^N \setminus [SI^\epsilon u < \lambda]) \geq \overline{D}(p, [SI^\epsilon u \geq \lambda]) \geq \overline{D}(p, Y) > 0$. Thus $\partial^M Y \subseteq \partial^M [SI^\epsilon u < \lambda] \ (\mathrm{mod}\, \mathcal{H}^{N-1})$.

Observe that, since $C \subseteq X$, we have that $SI^\epsilon(u)(x) \geq \lambda$ for all $x \in C$. Now, since $SI^\epsilon u \leq u$ and $u = \lambda$ on C, then $SI^\epsilon u = \lambda$ on C.

(ii) As in (i), let X be a connected component of $[u \geq \lambda]$ containing Y and Y'. Observe that $C \subseteq X$. Since $IS_\epsilon u \geq u$ and $X \subseteq [u \geq \lambda]$ we have

$$X \subseteq [IS_\epsilon u \geq \lambda]. \tag{31}$$

In particular, Y, Y' are classically connected inside $[IS_\epsilon u \geq \lambda]$.

Let us prove that Y, Y' are not M-connected in $[IS_\epsilon u \geq \lambda]$. We have the three obvious possibilities: either $\mathrm{Sat}(Y) \cap \mathrm{Sat}(Y') = \emptyset$, or $\mathrm{Sat}(Y) \subseteq \mathrm{Sat}(Y')$, or $\mathrm{Sat}(Y') \subseteq \mathrm{Sat}(Y) \ (\mathrm{mod}\, \mathcal{H}^N)$. Suppose that $\mathrm{Sat}(Y) \cap \mathrm{Sat}(Y') = \emptyset$. In this case, it is sufficient to prove that

$\partial^M \operatorname{SAT}(Y) \subseteq \partial^M [IS_\epsilon u \leq \lambda] \pmod{\mathcal{H}^{N-1}}$. Let $Z \in \mathcal{CC}^M([u \leq \lambda])$ be such that $\partial^M Z \cap \partial^M \operatorname{SAT}(Y) \neq \emptyset \pmod{\mathcal{H}^{N-1}}$. Assume that $\epsilon \in \{1/n : n \geq 1\}$ satisfies $\epsilon < |Z|$. Let $Z' \in \mathcal{CC}^M([u \leq \lambda])$ be such that $\partial^M Z' \cap \partial^M \operatorname{SAT}(Y) \neq \emptyset \pmod{\mathcal{H}^{N-1}}$. We prove that $Z' \subseteq [IS_\epsilon u \leq \lambda]$. Let $\lambda' > \lambda$, $p \in Z'$, and $R = \operatorname{cc}([u < \lambda'], p)$. Let us observe that Z is classically connected to Z' inside $[u < \lambda']$. Indeed, since $\partial^M \operatorname{SAT}(Y) \subseteq \partial \operatorname{SAT}(Y)$, then $\partial^M Z \cap \partial \operatorname{SAT}(Y) \neq \emptyset$ and $\partial^M Z' \cap \partial \operatorname{SAT}(Y) \neq \emptyset$. Since $\partial \operatorname{SAT}(Y) \subseteq \partial Y \subseteq [u = \lambda]$, then the set Z is classically connected to Z' inside $[u < \lambda']$. Hence $Z \subseteq R$, and, thus, $|R| > \epsilon$. We have that $R \in \mathcal{F}_\epsilon(u, p)$. Thus $IS_\epsilon u(p) \leq \lambda'$ for all $\lambda' > \lambda$. We obtain that $IS_\epsilon u(p) \leq \lambda$. In other words, $Z' \subseteq [IS_\epsilon u \leq \lambda]$ for all $Z' \in \mathcal{CC}^M([u \leq \lambda])$ such that $\partial^M Z' \cap \partial^M \operatorname{SAT}(Y) \neq \emptyset \pmod{\mathcal{H}^{N-1}}$. This implies that $\partial^M \operatorname{SAT}(Y) \subseteq \partial^M [IS_\epsilon u \leq \lambda]$. We conclude that Y and Y' are not M-connected inside $[IS_\epsilon u \geq \lambda]$.

Now we consider the case $\operatorname{Sat}(Y) \subseteq \operatorname{Sat}(Y')$, the other case being similar. Let $J^-(Y')$ be the internal M-boundary of Y' containing Y. Now, to prove that Y and Y' are not M-connected inside $[IS_\epsilon u \geq \lambda]$, it is sufficient to prove that $J^-(Y') \subseteq \partial^M [IS_\epsilon u \leq \lambda] \pmod{\mathcal{H}^{N-1}}$. The proof follows the same lines as the above one with $J^-(Y')$ in place of $\partial^M \operatorname{SAT}(Y)$. Indeed, it can be reduced to it by an inversion in $\mathbb{R}^N$ that sends a point in the hole of Y' containing $\operatorname{Sat}(Y)$ to infinity. Just observe that $u = \lambda$ on $J^-(Y')$.

Let us prove that $IS_\epsilon u = \lambda$ on C. First, observe that, by Proposition 2, $\lambda = u(p) \leq IS_\epsilon u(p)$, for all $p \in C$. Without loss of generality, by doing an inversion in $\mathbb{R}^N$ if necessary, we may assume that $\operatorname{Sat}(Y) \cap \operatorname{Sat}(Y') = \emptyset \pmod{\mathcal{H}^N}$. Let $p \in C$. Let $Z \in \mathcal{CC}^M([u \leq \lambda])$ be such that $\partial^M Z \cap \partial^M \operatorname{SAT}(Y) \neq \emptyset \pmod{\mathcal{H}^{N-1}}$. By Lemma 12, we may assume that Z is closed and connected. Let $\epsilon \in \{1/n : n \geq 1\}$, $\epsilon < |Z|$. Let $\lambda' > \lambda$, $R = \operatorname{cc}([u < \lambda'], p)$. Since $\partial^M \operatorname{SAT}(Y) \subseteq \partial \operatorname{SAT}(Y) \subseteq C \subseteq [u = \lambda]$, then Z is connected to p inside $[u < \lambda']$. Hence, $Z \subseteq R$ and, thus, $|R| > \epsilon$. We have that $R \in \mathcal{F}_\epsilon(u, p)$. We obtain that $IS_\epsilon u(p) \leq \lambda'$ and this is true for all $\lambda' > \lambda$. We have then $IS_\epsilon u(p) \leq \lambda$. Then $IS_\epsilon u(p) = \lambda$ for all $p \in C$. □

Remark 8. The argument above can be repeated for a finite number of M-boundaries of Y and Y' but not for all the measure theoretic boundary, because there may be a countable family of sets $Z \in \mathcal{CC}^M([u \leq \lambda])$ such that $\partial^M Z \cap \partial^M \operatorname{SAT}(Y) \neq \emptyset \pmod{\mathcal{H}^{N-1}}$ and $\epsilon > 0$ cannot work for all of them. It may be small indecomposable sets of $[u \leq \lambda]$ bounding Y and/or Y'.

Lemma 20. *Let $v \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that v is constant in a neighborhood of $\partial\Omega$. Let $\lambda \in (\inf_\Omega v, \sup_\Omega v)$ be such that $[v \geq \lambda]$ is a set of finite perimeter and $|[v = \lambda]| = |[IS_\epsilon v = \lambda]| = 0$ for any $\epsilon \in \{1/n : n \geq 1\}$. Let $Y, Y' \in \mathcal{CC}^M([v \geq \lambda])$, $Y \neq Y'$ be classically connected inside $[v \geq \lambda]$. We assume that Y and Y' are the closures of sets with all points of density 1 and $|\partial Y| = |\partial Y'| = 0$. Suppose that Y is classically connected to Y' inside $[v \geq \lambda]$ by a continuum $C \subseteq [v = \lambda]$ such that $C \supseteq Q(Y) \cup Q(Y')$. Let $w = IS_\epsilon v$, $x \in C$. Then $\eta_+(w, x, \lambda) = \lambda$ for $\epsilon \in \{1/n : n \geq 1\}$ be small enough.*

Proof: Recall that the sets Y, Y' are closed and connected and satisfy $\partial Y, \partial Y' \subseteq [v = \lambda]$. Moreover, by Lemma 19, ii), $IS_\epsilon v = \lambda$ on C. Since $|[v = \lambda]| = 0$, we have $|Y| = |\{z \in Y : v(z) > \lambda\}|$. Let $\alpha > \lambda$ be such that $|Y \cap [v \geq \alpha]| > 0$, $|Y' \cap [v \geq \alpha]| > 0$ and $[v \geq \alpha]$ is a set of finite perimeter. Let Y_α, Y'_α be (classical) components of $Y \cap [v \geq \alpha]$, $Y' \cap [v \geq \alpha]$ of > 0 measure. Note that $Y_\alpha, Y'_\alpha \subseteq Y$ and $Y_\alpha = \mathrm{cc}([v \geq \alpha], Y_\alpha)$, $Y'_\alpha = \mathrm{cc}([v \geq \alpha], Y'_\alpha)$. Let $\epsilon \in \{1/n : n \geq 1\}$ be such that $|Y_\alpha|, |Y'_\alpha| > \epsilon$.

Let $\lambda < \mu \leq \alpha$. Let $Y_\mu = \mathrm{cc}([v \geq \mu], Y_\alpha)$, $Y'_\mu = \mathrm{cc}([v \geq \mu], Y'_\alpha)$. Observe that $Y_\mu \subseteq Y$, $Y'_\mu \subseteq Y'$. Since $\mu \leq v \leq IS_\epsilon v$ on $Y_\mu \cup Y'_\mu$, the sets $Y^\epsilon_\mu := \mathrm{cc}([IS_\epsilon v \geq \mu], Y_\mu)$, $Y^{\epsilon,\prime}_\mu := \mathrm{cc}([IS_\epsilon v \geq \mu], Y'_\mu)$ contain Y_μ, resp. Y'_μ, for every $\lambda < \mu \leq \alpha$. Observe that, since $Y_\alpha \subseteq Y_\mu$, then also $Y^\epsilon_\alpha \subseteq Y^\epsilon_\mu$.

By symmetry, without loss of generality, we may assume that either $\mathrm{Sat}(Y) \cap \mathrm{Sat}(Y') = \emptyset$ or $\mathrm{Sat}(Y) \subseteq \mathrm{Sat}(Y')$ $(\mathrm{mod}\,\mathcal{H}^N)$. In any case $Y \subset\subset \Omega$. Let $x \in C$. Suppose that $\eta_+(w, x, \lambda) > \lambda$ where $\eta_+(w, ., .)$ denotes the η_+, defined in Section 2, corresponding to w. Let $\lambda < \mu < \hat{\mu} < \eta < \eta_+(w, x, \lambda)$. Let $X_{\lambda,\epsilon,x} = \mathrm{cc}(\{z \in \overline{\Omega} : w(z) \in [\lambda, \eta_+(w, x, \lambda)]*\}, x)$. Let $X_{\lambda,\eta} = \{z \in X_{\lambda,\epsilon,x} : \lambda \leq w(z) \leq \eta\}$ (a connected set). Note that $\{z \in X_{\lambda,\epsilon,x} : \mu \leq w(z) \leq \hat{\mu}\}$ is a nonempty set.

Since $Y_\mu \subseteq Y$ and $Y^\epsilon_\mu = \mathrm{cc}([w \geq \mu], Y_\mu)$, we have that $Y \cap Y^\epsilon_\mu \neq \emptyset$. Hence, $Y \cup Y^\epsilon_\mu$ is connected. Let $p_0 \in \partial^M \mathrm{SAT}(Y)$, $p_1 \in Y^\epsilon_\mu$, and let K be a continuum contained in $Y \cup Y^\epsilon_\mu$ joining p_0 and p_1.

Now, we claim that there is a point $p \in Y \cap X_{\lambda,\eta}$ such that $\mu \leq w(p) \leq \hat{\mu}$. Let $L_0 = \{y \in K : w(y) < \mu\}$, $L_1 = \{y \in K : w(y) > \hat{\mu}\}$. Observe that both are open sets in K and $L_0 \subseteq Y$. Since $w(p_0) = \lambda < \mu$, and $w(p_1) \geq \mu$, then $p_0 \in L_0$, $p_1 \notin L_0$. Then L_0 is a neighborhood of p_0 in K. We observe that $L_0 \subseteq [\lambda \leq w \leq \eta]$. Indeed, since $L_0 \subseteq K \subseteq Y \cup Y^\epsilon_\mu$, then $w(y) \geq \lambda$ for all $y \in L_0$, and, on the other hand, $w(y) < \mu < \eta$, for all $y \in L_0$. Given $k \geq 1$, there is a finite sequence of points $p^k_0, p^k_1, \dots, p^k_{N_k}$ in K with $p^k_0 = p_0$, $p^k_{N_k} = p_1$, and $d(p^k_i, p^k_{i+1}) < \frac{1}{k}$. Let j_k be the first index such that $p^k_i \in L_0$ and $p^k_{i+1} \notin L_0$. Observe that $j \leq N_k - 1$. Since K is a compact set, we may assume that $p^k_{j_k} \to p$ as $k \to \infty$. Then, also $p^k_{j_k+1} \to p$. Since $w(p^k_{j_k}) < \mu$ and $w(p^k_{j_k+1}) \geq \mu$, we have that $w(p) = \mu$. On the other hand, $p \in \overline{L}_0 \subseteq [\lambda \leq w \leq \eta]$, and, being limit of points not in L_0, then $p \neq p_0$. Let $m \geq 1$, and let $k_0 \geq m$ be such that $|p^k_{j_k} - p| < \frac{1}{m}$ for all $k \geq k_0$. Recall that $p^k_i \in L_0$ for all $i \leq j_k$ and $d(p^k_i, p^k_{i+1}) \leq \frac{1}{k} \leq \frac{1}{m}$ for all i. Let γ_k be the poligonal joining p^k_i to p^k_{i+1} for all $0 \leq i \leq j_k$. Then $\sup_{p\in\gamma_k} d(p, \overline{L}_0) < \frac{1}{m}$. Letting $m \to \infty$, and taking a subsequence, if necessary, we may assume that γ_k converges to some continuum $\gamma \subseteq \overline{L}_0 \subseteq Y$ joining p_0 to p ([**14**, Theorems 3.16 and 3.18], [**22**, vol. II, p. 111]). We conclude that $p \in Y \cap X_{\lambda,\eta}$. In a similar way we obtain that there is some $q \in Y' \cap X_{\lambda,\eta}$ such that $\mu \leq w(q) \leq \hat{\mu}$. Here we work with the M-boundary $J(Y')$, being $J(Y') = \partial^M \operatorname{SAT}(Y')$, if $\operatorname{Sat}(Y) \cap \operatorname{Sat}(Y') = \emptyset$, or $J(Y')$ being the external M-boundary of the hole of Y' containing Y (thus, an internal boundary of Y').

Summarizing, we have shown that the sets $Y^\epsilon_{\mu,\hat{\mu}} := \{z \in Y \cap X_{\lambda,\eta} : \mu \leq w(z) \leq \hat{\mu}\}$, $Y^{\epsilon,\prime}_{\mu,\hat{\mu}} := \{z \in Y' \cap X_{\lambda,\eta} : \mu \leq w(z) \leq \hat{\mu}\}$ are non empty. Suppose that $Y^\epsilon_{\mu,\hat{\mu}}$ and $Y^{\epsilon,\prime}_{\mu,\hat{\mu}}$ are classically connected inside $X_{\lambda,\epsilon,x} \cap U_{\mu,\hat{\mu}}[w]$. Let Q be a continuum connecting $Y^\epsilon_{\mu,\hat{\mu}}$ to $Y^{\epsilon,\prime}_{\mu,\hat{\mu}}$ and contained in $X_{\lambda,\epsilon,x} \cap U_{\mu,\hat{\mu}}[w]$. Observe that $w \geq \mu$ on Q. Thus $Q \subseteq \{p \in \overline{\Omega} : w > \lambda\}$ which is an open set in $\overline{\Omega}$, because $IS_\epsilon v$ is continuous in $\overline{\Omega}$. Thus, Y would be classically connected and also M-connected to Y' inside $[IS_\epsilon v \geq \lambda]$ by an open set. In contradiction with the fact that Y, Y' are contained in two different indecomposable sets of $[IS_\epsilon v \geq \lambda]$, a fact proved in Lemma 19. Thus $Y^\epsilon_{\mu,\hat{\mu}}$ and $Y^{\epsilon,\prime}_{\mu,\hat{\mu}}$ are not connected inside $X_{\lambda,\epsilon,x}$, a contradiction since this last set is a monotone section of the topographic map of w. □

Proof of Theorem 6: By Lemma 13, we know that $SI^\epsilon u \in WBV(\Omega) \cap C(\overline{\Omega})$. Moreover, if $\epsilon < |R(\partial\Omega)|$, then $SI^\epsilon u = u = \text{constant in } R(\Omega)$. Let $\lambda \in (\inf_\Omega u, \sup_\Omega u)$ be such that $[u \geq \lambda]$, $[SI^\epsilon u \geq \lambda]$ are sets of finite perimeter, for every $\epsilon \in \{1/n : n \geq 1\}$, and $|[u = \lambda]| = |[SI^\epsilon u = \lambda]| = |[IS_\delta SI^\epsilon = \lambda]| = 0$ for every $\epsilon, \delta \in \{1/n : n \geq 1\}$. We have thus excluded a set of levels of measure zero. Assume that λ is a critical level of u. By Lemma 18 there exist $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$ which are classically connected inside $[u \geq \lambda]$ by a continuum $C \supseteq Q(Y) \cup Q(Y')$, $C \subseteq [u = \lambda]$. By Lemma 19, i), $Y, Y' \in \mathcal{CC}^M([SI^\epsilon u \geq \lambda])$ and are classically connected in $[SI^\epsilon u \geq \lambda]$ by $C \subseteq [SI^\epsilon u = \lambda]$. Thus, for ϵ small enough, the assumptions of Lemma 20 are satisfied. Hence $\eta_+(IS_\delta SI^\epsilon u, x, \lambda) = \lambda$ for $x \in C$ and $\epsilon, \delta \in \{1/n : n \geq 1\}$ small enough. Since these levels form a countable set, the theorem is proved. □

Lemma 21. *Let $u \in WBV(\Omega) \cap C(\overline{\Omega})$. Suppose that u is constant in a neighborhood of $\partial\Omega$. Let $\lambda \in (\inf_\Omega u, \sup_\Omega u]$ be such that $[u \geq \lambda]$ is a set of finite perimeter and $|[u = \lambda]| = 0$.*

(i) *Let $X \in \mathcal{CC}^M([u \leq \lambda])$ and let $\{J\} = \mathcal{E}^M(X)$ (resp. $J \in \mathcal{I}^M(X)$). Suppose that $J \notin \mathcal{I}^M([u \geq \lambda])$ (resp. $J \notin \mathcal{E}^M([u \geq \lambda])$). Then there exist $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$, $Y \neq Y'$, which are classically connected by a continuum $C \subseteq [u = \lambda]$.*

(ii) *Let $X \in \mathcal{CC}^M([u \geq \lambda])$ and let $\{J\} = \mathcal{E}^M(X)$ (resp. $J \in \mathcal{I}^M(X)$). Suppose that $J \notin \mathcal{I}^M([u \leq \lambda])$ (resp. $J \notin \mathcal{E}^M([u \leq \lambda])$). Then there exist $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$, $Y \neq Y'$, which are classically connected by a continuum $C \subseteq [u = \lambda]$, where $Z \in \mathcal{CC}^M([u \leq \lambda])$.*

Proof: i) Let $X \in \mathcal{CC}^M([u \leq \lambda])$ and let $\{J\} = \mathcal{E}^M(X)$. If $J \notin \mathcal{I}^M([u \geq \lambda])$ then there exist $Y, Y' \in \mathcal{CC}^M([u \geq \lambda])$, $Y \neq Y'$, such that $H^{N-1}(J \cap \partial^M Y) > 0$, $H^{N-1}(J \cap \partial^M Y') > 0$. Since $J = \partial^M \operatorname{SAT}(X) \subseteq \partial \operatorname{SAT}(X) \subseteq [u = \lambda]$ and $\partial \operatorname{SAT}(X)$ is a continuum, then Y and Y' are classically connected inside $[u \geq \lambda]$. If $J \in \mathcal{I}^M(X)$ but $J \notin \mathcal{E}^M([u \geq \lambda])$ the same argument proves the result.

ii) Let $X \in \mathcal{CC}^M([u \geq \lambda])$ and $\{J\} = \mathcal{E}^M(X)$. Suppose that $J \notin \mathcal{I}^M([u \leq \lambda])$. Then there exist $Z, Z' \in \mathcal{CC}^M([u \leq \lambda])$ (not equal) such that $H^{N-1}(J \cap \partial^M Z) > 0$, $H^{N-1}(J \cap \partial^M Z') > 0$. Since $J = \partial^M \operatorname{SAT}(X) \subseteq \partial \operatorname{SAT}(X) \subseteq [u = \lambda]$ and $\partial \operatorname{SAT}(X)$ is a continuum, then Z and Z' are classically connected inside $[u \leq \lambda]$ by $\partial \operatorname{SAT}(X) \subset\subset \Omega$. If $J \in \mathcal{I}^M(X)$ but $J \notin \mathcal{E}^M([u \leq \lambda])$ then we arrive at the same conclusion with the same argument.

Let $J(Z)$ be the M-boundary of the (external or internal) hole of Z, $J(Z) \in \mathcal{E}^M(Z) \cup \mathcal{I}^M(Z)$ such that $H^{N-1}(J \cap J(Z)) > 0$. If $J(Z) \subseteq J$, then $J(Z) = J$ since both are boundaries of simple sets ([**2**, Proposition 7]). Since $H^{N-1}(\partial^M Z' \cap J) > 0$ this would imply that $H^{N-1}(\partial^M Z \cap \partial^M Z') > 0$, a contradiction. Hence, $J(Z)$ is not contained in J. Thus there are M-connected components $Y = X$ and Y' of $[u \geq \lambda]$ such that $H^{N-1}(J(Z) \cap \partial^M Y') > 0$ (recall that also $H^{N-1}(J(Z) \cap \partial^M Y) > 0$). Now, observe that $J(Z) = \partial^M \operatorname{SAT}(Z)$ if $J(Z) \in \mathcal{E}^M(Z)$, and $J(Z) = \partial^M \operatorname{SAT}(Z, H)$ if $J(Z) \in \mathcal{I}^M(Z)$ coincides with the external boundary of an M-hole H of Z, where $\operatorname{SAT}(Z, H)$ denotes the saturation of Z with the hole H taken as unbounded M-component of its complement. In the first case, we let $C = \partial \operatorname{SAT}(Z)$, in the second case, $C = \partial \operatorname{SAT}(Z, H)$. Since C is a continuum and $J(Z) \subseteq C$, then Y can be classically connected to Y' by a continuum $C \subseteq [u = \lambda]$. □

Proof of Proposition 7: Let Λ_1 be the set of $\lambda \in \mathbb{R}$ such that $[u \geq \lambda]$, $[SI^\epsilon u \geq \lambda]$ are sets of finite perimeter, and $|[u = \lambda]| = |[SI^\epsilon u = \lambda]| = |[IS_\delta SI^\epsilon u = \lambda]| = 0$, for all $\epsilon, \delta \in \{1/n : n \geq 1\}$. Observe that $|\mathbb{R} \setminus \Lambda_1| = 0$. Let Φ be the set of $\lambda \in \Lambda_1$ such that $\eta_+(IS_\delta SI^\epsilon u, x, \lambda) = \lambda$ for some $x \in \overline{\Omega}$ and some $\epsilon, \delta \in \{1/n : n \geq 1\}$. By Theorem 2, Φ is a countable set. Thus $\Lambda := \Lambda_1 \setminus \Phi$ satisfies $|\mathbb{R} \setminus \Lambda| = 0$. By Lemma 21, Lemma 19, and Lemma 20, if $\lambda \in \Lambda$, we have $\mathcal{E}^M([u \geq \lambda]) = \mathcal{I}^M([u < \lambda])$ and $\mathcal{I}^M([u \geq \lambda]) = \mathcal{E}^M([u < \lambda])$. $\square$

Remark 9. Looking for a connected filter which conmutes with contrast inversion, S. Masnou [**26**] introduced the notion of grain filter proving that it is a morphological filter which conmutes with contrast inversion, i.e., with the map which associates to an image u the image $-u$, in case of smooth images (indeed for N-times differentiable images defined on $\mathbb{R}^N$). Since the grain filter is stable with respect to the supremum norm (indeed it satisfies properties ii) and iv) of Proposition 2), by a limit process, his result immediately implies that the grain filter can be extended to continuous functions and it commutes with contrast inversion. Such commutation result could also be deduced Theorem 7 but we shall not pursue this here. It has also been deduced by P. Monasse [**30**] in a different framework, when the filter is defined on continuous functions using the classical notion of connectedness. By the results of this paper both definitions of the grain filter coincide for continuous functions.

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