U.S. GDP and S&P 500

An Inquiry into the Nature and Causes of the Econometric Relation between GDP and Stock Market in the U.S.

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ABSTRACT

We are made to believe the stock market is a good measure for the actual economy. However, while the S&P 500 has been strong for almost 40 years, U.S. GDP growth has slowed down considerably. This paper aims to conduct an analysis of the established econometric relationship between these economic concepts and delve into its joint dynamics and linear interdependencies. To do this we use a VAR model to test Granger causality and analyze the Impulse Response Functions based on a quarter time series data for the year 1947 to 2018. While the S&P 500 cannot be predicted in our study, we can show a positive historical relation between current U.S. GDP growth and past S&P 500 growth when comparing our results with an AR model. As such, our analysis aims to utilize S&P 500 as a short-run leading indicator to forecast U.S. GDP behavior.
INTRODUCTION

The U.S. Gross Domestic Product

The Gross Domestic Product (GDP) is the monetary measure of the market value of all the final goods and services produced within a country’s border in a given period. There are three ways to calculate GDP: the income approach, the production approach and the expenditure approach. For the purposes of this study, we used the expenditure approach to calculate the GDP because best relates with the data used in our analysis. The U.S. has the highest GDP in the world. Consumer spending represents over two-thirds of U.S. GDP, while government budget and business investment make up an estimated 20% and 15% of U.S. GDP respectively. The U.S. current account is currently at a 3% deficit.

The Stock Market Index in the U.S.

The Stock Market is a financial market in which people trade business shares and/or stocks. The Stock Market Index measures the performance of the stock market in a given country. There are different mathematical methods to compute Stock Market Indexes, but for the purposes of our inquiry, this study will be based on a Weighted Average Market Capitalization index because the movements depend on market capitalization. The most significant weighted-average-market-capitalization index in the U.S. is the Standard & Poor's 500 or S&P 500 (Kenton and Murphy, 2019). It is represented by the 500 largest U.S. assets by market capitalization and is considered the best representation of the U.S stock market.

Justification of the project

As Nobel prize-winning economist Robert J. Shiller has said: “I have always wondered myself, why do we have to hear this every night, what the stock market is doing.” Stock market information is constantly being reported on major American news channels 24/7. Economists say it is because there is a strong relation between the stock market and the health of the American economy (Campbell, 1987; Cochrane, 1991; Levine and Zervos, 1998). However, some people started claiming a divergence from typical growth patterns from the last decade (Scheltens, 2018; Herrmann, 2019). According to the World Bank, the average American economy growth rate in the last decade slowed down 1.5-2% of difference from typical patterns while stock markets growth increased 4% of difference. All in all, what is the relationship between stock markets and the economy? Are stock markets an indicator for the economy and/or vice versa? What happens to the economy when there is a shock to the stock markets and vice versa?
The paper addresses the historical econometric relationship between the U.S. GDP and the S&P 500, as the main representations of the American actual economy and stock market. The aim is to infer into its existing historical relationship and quantify the importance of linear interdependences and shocks for quarter periods between 1947 and 2018.

We start by separately analyzing the persistence of U.S. GDP and S&P 500 time series (The Autocorrelation Function (ACF)) and quantifying its linear dependences with previous values (The Auto-Regressive model (AR)). We out-of-sample forecast the independent series to evaluate the accuracy of the performance (AR Forecasting). Next, we include linear interdependencies between U.S. GDP and S&P 500 to infer into its econometric relationship (The Vector Auto-Regressive model (VAR)). We test whether S&P 500 can be used as an indicator to forecast U.S. GDP and vice versa (Granger Causality), and we study the effects of a shock in the S&P 500 on the U.S. GDP and vice versa (Impulse Response Functions). Finally, we use the joint dynamics established in preceding sections to out-of-sample forecast and compare results with the previous forecasted outcomes to quantify the improvement when including indicators (VAR Forecasting).

**LITERATURE REVIEW**

In a Robinson Crusoe economy, stock market growth increase should exactly match real GDP growth (Herrmann, 2019). The finance-growth hypothesis (Schumpeter, 1932; McKinnon, 1973) states that whether there is economic growth, there is earnings growth that causes adjustments in stock prices. However, the modern economy is unlikely to operate under the same assumptions. Thus, there have been several attempts to quantify the link between the stock market and economic growth. McMillan (2005) examined whether there exists a relationship between S&P 500 and American industrial production. He used a Vector Error Correction Model to test for cointegration between integrated time series and concluded that stock prices have a short-run predictability on industrial production. Contrarily, Siliverstovs and Duong (2006) found that stock markets exerts a weak influence on GDP that is barely statistically detectible in Europe. They addressed a VAR relation between the stock market and real GDP for five European countries. Similarly, MSCI (2010) conducted a linear regression to conclude that long term real stock returns fell behind long term GDP growth in many countries. They inferred that global markets, the role of enterprise, and expectations are factor causes of such discrepancies. Furthermore, Ritter (2005) found a negative correlation of real stock returns and per capita GDP growth over 1900–2002 in sixteen countries representing 90% of world market capitalization. He argues that if economic growth is due to an increase in technological change, then shareholders and stock markets does not necessarily benefit. On the whole, research in the field varies in methods and results.
METHODOLOGY

The Autocorrelation Function (ACF)

The Autocorrelation Function (ACF) is a correlation function between two values of the same variable $X$ at different times. Thus, the ACF is a linear relationship between observations separated by $t$ time lags and are values between -1 and 1, with 1 being a perfect positive correlation between two periods. We often expect a positive ACF when time lags are small. ACF converges toward 0 when there are greater time differences between observations, although it depends on the series persistence.

$$
\text{autocorr}(X, t) = corr[X, \text{lag}(X, t)] = \frac{cov[X, \text{lag}(X, t)]}{\text{var}(X)}
$$

The Auto-Regressive model (AR)

The Auto-Regressive (AR) model regresses a value from a time series on previous values from that same time series. The response variable is the predictor, the order is the number of lags, and the error term follows the same assumptions than in a linear regression model. Thus, the AR model can predict future behavior based on past behavior. There is a clear relation between the ACF and the AR models: the greater the Autocorrelation Function of a variable, the greater the predictors coefficients in the Auto-Regressive model.

$$
X_t = \beta_0 + \beta_1 \times X_{t-1} + \cdots + \beta_p \times X_{t-p} + \epsilon_t
$$

Forecasting and Mean-Squared-Forecast-Errors (MSFE)

Out-of-sample forecasting is the behavioral study of forecasted outcomes given the performance of a regressive model (Diebold and Rudebusch, 1991). It is based in the following scenario: imagine it is the fourth quarter of 1971 and we need a forecast of U.S. GDP or the S&P 500 for the next quarter. We use a model to forecast, but only using data available until Q4-1971. Next, say it is the first quarter of 1972 and we make the forecast for the second quarter of 1972, using the data that is now available. We continue to do this until the last observation, and we get a series of forecasts for the model.
A quantitative method to judge the out-of-sample forecasting is to compute the mean-squared-forecast-errors (MSFE). It quantifies the difference between the forecast outcome and the actual outcome. Thus, the higher the MSFE, the worse the model performed in the out-of-sample period.

\[
\text{MSFE} = \frac{1}{M} \sum_{m=1}^{M} (X_m - \hat{X}_m)^2
\]

Where:
- \(X_m\) = actual observation of \(m\)
- \(\hat{X}_m\) = estimation of the observation \(m\)
- \(M\) = total number of observations

**The Vector Auto-Regressive model (VAR)**

The most comprehensive model to analyze the econometric relation between these two variables is by modeling a Vector Auto-Regressive. VAR models are used to capture joint dynamics and we are interested in the linear interdependencies between U.S. GDP and the S&P 500 time series.

VAR models fit a multivariate time-series regression of each dependent variable on lags of itself and of the other dependent variables. Thus, the evolution of each variable is based on its own lagged values, the lagged values of the other variable and an error term:

\[
\begin{align*}
GDP_t &= \beta_{1,11} GDP_{t-1} + \cdots + \beta_{p,11} GDP_{t-p} + \beta_{1,12} SP500_{t-1} + \cdots + \beta_{p,12} SP500_{t-p} + \epsilon_{1,t} \\
SP500_t &= \beta_{1,21} SP500_{t-1} + \cdots + \beta_{p,21} SP500_{t-p} + \beta_{1,22} GDP_{t-1} + \cdots + \beta_{p,22} GDP_{t-p} + \epsilon_{2,t}
\end{align*}
\]

VAR models are used when there are few variables because parameters increase proportionally with the number of variables. U.S. GDP and S&P 500 limit the number of parameters, causing VAR models to be ideal for such econometric relationships. We use matrices to compactly show VAR models:

\[
\begin{bmatrix}
GDP_t \\
SP500_t
\end{bmatrix} =
\begin{bmatrix}
\beta_{1,11} & \beta_{1,12} \\
\beta_{1,21} & \beta_{1,22}
\end{bmatrix}
\begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix} + \cdots +
\begin{bmatrix}
\beta_{p,11} & \beta_{p,12} \\
\beta_{p,21} & \beta_{p,22}
\end{bmatrix}
\begin{bmatrix}
GDP_{t-p} \\
SP500_{t-p}
\end{bmatrix} +
\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t}
\end{bmatrix}
\]

To determine the order of the VAR model, we considered the Akaike’s Information Criteria (AIC), the Bayesian Information Criteria (BIC) and the method summarized in FIGURE 1. This method considers any number of lags for our VAR model, continuously rejects the inefficient outcomes, and ends up by selecting the optimal order for the VAR model. The method is based on the coefficient matrix of lag \(p\):
Coefficient matrix of lag $p = \begin{bmatrix} \beta_{p,11} & \beta_{p,12} \\ \beta_{p,21} & \beta_{p,22} \end{bmatrix}$

The idea is we start with $p$ lags; a VAR ($p$) model. If the coefficient matrix of lag $p$ is not statistically significant at the 5% significance level\(^1\), we conclude that the optimal number of lags is $p-1$ where the coefficient matrix of the last lag is statistically significant at the 5% significance level. Otherwise, we proceed until we find the maximum number of lags that is statistically significant with a coefficient matrix.

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\(^1\) Note: the coefficient matrix of lag $p$ is not statistically significant if none of its coefficients is statistically significant at the 5% significance level.
power, the MSFE would be smaller when we include them as explanatory variables. Alternatively, they are exogenous in the time series with respect to the other variable.

For \( s > 0 \), time series S&P 500 fails to Granger-cause time series GDP if:

\[
MSFE[\text{Proj}(GDP_{t+s}|GDP_t, GDP_{t-1}, ...)] = MSFE[\text{Proj}(GDP_{t+s}|GDP_t, GDP_{t-1}, ..., SP500_t, SP500_{t-1}, ...)]
\]

Where \( \text{Proj}(GDP_{t+s}|GDP_t, GDP_{t-1}, ...) \) denotes the best linear prediction for GDP.

For \( s > 0 \), time series GDP fails to Granger-cause time series S&P 500 if:

\[
MSFE[\text{Proj}(SP500_{t+s}|SP500_t, SP500_{t-1}, ...)] = MSFE[\text{Proj}(SP500_{t+s}|SP500_t, SP500_{t-1}, ..., GDP_t, GDP_{t-1}, ...)]
\]

Where \( \text{Proj}(SP500_{t+s}|SP500_t, SP500_{t-1}, ...) \) denotes the best linear prediction for SP500.

**Impulse Response Functions (IRFs)**

Impulse Response Functions (Pesaran and Shin, 1998) summarize the effect of a random shock in one variable to another variable on an interval of time. This helps us understand the dynamics of the VAR model and to study the related joint effects of S&P 500 and GDP time series: if a shock \( \varepsilon_{it} \) in the impulse variable time series can be observed in the response variable, there is a predictable relation between both.

If we assume a stationary time series, the VAR \( (p) \) model can be written in the following matrix equation using geometric series properties:

\[
\lim_{d \to \infty} \begin{bmatrix} GDP_{gt} \\ SP500_{gt} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} + \Phi_0 \ast \begin{bmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{bmatrix} + \Phi_1 \ast \begin{bmatrix} \varepsilon_{1,t-2} \\ \varepsilon_{2,t-2} \end{bmatrix} + \cdots + \Phi_d \ast \begin{bmatrix} \varepsilon_{1,t-d} \\ \varepsilon_{2,t-d} \end{bmatrix}
\]

The VAR \( (p) \) model is expressed in terms of past shocks; the effects of shocks \( \varepsilon_{i,s} \) are captured by \( \Phi_s \). The coefficient \( \Phi_{t-s} \) is bigger than \( \Phi_{t-s-1} \) for any \( t \) and \( s \), because of the stationary assumption. This gives the first residuals a high relative importance.

\[
\frac{\partial \begin{bmatrix} GDP_{gt} \\ SP500_{gt} \end{bmatrix}}{\partial \varepsilon_{i,s}'} = \Phi_s
\]

\[2\] Note: see section Appendix for the mathematical approach.
DATA

Data resource

U.S. Gross Domestic Product

The data source used in this paper is provided by the OECD, which provides accurate data from 1947. The study is based on a quarterly database from Q2-1947 to Q3-2018, which computes a total of 287 observations.

The accuracy of the estimators is limited by the sample. A large sample provides more accurate and precise coefficients (Stock and Watson, 2011). The quarterly database from OECD is the largest and most reliable sample available.

The inquiry focuses on the real growth rate of U.S. GDP. While the GDP level on its own does not give us useful information, the change in GDP has important economic connotations, as it can relate with business cycles and has a stationary time series.

The Stock Market Index in the U.S.: The S&P 500


The object time period relies on quarterly data from Q2-1947 to Q3-2018. We are interested in the Average Price per Share in Month Ending Quarter. Accordingly, we will consider the months March, June, September and December from 1947 to 2018 in Robert J. Shiller’s database.

We are interested in the growth rate of the S&P 500. The level itself has little meaning, but the changes in S&P 500 level over a period has economic connotations that relate to the business cycle.

Data description

The U.S. Gross Domestic Product

\( t \) denotes the set of quarter time series:
$t = \{1947-\text{Q}2, 1947-\text{Q}3, \ldots, 2018-\text{Q}2, 2018-\text{Q}3\}$

$\text{GDP}_{gt}$ denotes the real growth of U.S. GDP at quarter $t$ compared with the previous quarter $t-1$:

$$\text{GDP}_{gt} = \text{real growth of the U.S. GDP at } t \text{ compared with } t-1$$

*FIGURE 2 – U.S. GDP growth ($\text{GDP}_{gt}$)*

The time series of $\text{GDP}_{gt}$ satisfies the following stationary conditions:

1. $E(\text{GDP}_{gt}^2) < \infty$
2. $E(\text{GDP}_{gt}) = m$, with $m$ constant
3. $\text{Cov}(\text{GDP}_{gt+h}, \text{GDP}_{gt}) = \text{Cov}(\text{GDP}_{s}, \text{GDP}_{s+h})$ for all $s, t$

*The Stock Market Index in the U.S.: The S&P 500*

$t$ denotes the set of quarter time series:

$$t = \{1947-\text{Q}2, 1947-\text{Q}3, \ldots, 2018-\text{Q}2, 2018-\text{Q}3\}$$

$M_t$ denotes the Average Price per Share in Month Ending Quarter:

$$M_t = \text{Monthly Average Price per Share at } t$$

$\text{CPI}_t$ denotes the consumer price index at quarter $t$: [Diagram of CPI over time]
CPI\textsubscript{t} = consumer price index at \( t \)

We can get the real value of the index at quarter \( t \) by multiplying \( M\textsubscript{t} \) times CPI\textsubscript{t} and chaining it into a dollar base year. We chain 1983 dollar as the base year, which is the 145\textsuperscript{th} quarter. Therefore:

\[
P\textsubscript{t} = \text{Real Monthly Average Price per Share at } t \text{ (chained 1983 dollar)}
\]

\[
P\textsubscript{t} = M\textsubscript{t} \times \frac{\text{CPI}\textsubscript{t}}{\text{CPI}\textsubscript{145}}
\]

**FIGURE 3 – Prices (\( P\textsubscript{t} \))**

Prices time series (\( P\textsubscript{t} \)) are non-stationary and behave like a random walk (Fama, 1970). However, the growth rate of the S&P 500 index has a stationary time series and we need it to compute the proper econometric match with the growth rate of the U.S. GDP.

The growth rate is the return between two periods and can be computed using the property of natural logarithms. SP500\textsubscript{g\textsubscript{t}} denotes the log-return of the index at quarter \( t \) compared with the previous quarter \( t\text{-}1 \):

\[
\text{SP500}\textsubscript{g\textsubscript{t}} = \text{real growth of the S&P 500 at } t \text{ compared with } t\text{-}1
\]

\[
\text{SP500}\textsubscript{g\textsubscript{t}} = \log P\textsubscript{t} - \log P\textsubscript{t\text{-}1}
\]
The time series of \( SP500g_t \) satisfies the following stationary conditions:

1. \( E(SP500g_t^2) < \infty \)
2. \( E(SP500g_t) = m \), with \( m \) constant
3. \( \text{Cov}(SP500g_{t+h}, SP500g_t) = \text{Cov}(SP500g_s, SP500g_{s+h}) \) for all \( s, t \)

**Joint Data Description**

\( GDPg_t \) and \( SP500g_t \)

\( t \) denotes the set of quarter time series:

\[ t = \{1947-Q2, 1947-Q3, \ldots, 2018-Q2, 2018-Q3\} \]

\( GDPg_t \) denotes the real growth of U.S. GDP at quarter \( t \) compared with the previous quarter \( t-1 \):

\[ GDPg_t = \text{real growth of the U.S. GDP at } t \text{ compared with } t-1 \]

\( SP500g_t \) denotes the real growth of the S&P 500 of the index at quarter \( t \) compared with the previous quarter \( t-1 \):

\[ SP500g_t = \text{real growth of the S&P 500 at } t \text{ compared with } t-1 \]
The study is based on 287 quarterly periods from Q2-1947 to Q3-2018. For each quarter $t$, the study relies on a match between an observation of $GDP_g$ and an observation of $SP500_g$. This computes a data base with a total of 574 observations.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GDP_g$</td>
<td>287</td>
<td>.7826253</td>
<td>.9416798</td>
<td>-2.595647</td>
<td>3.930121</td>
</tr>
<tr>
<td>$SP500_g$</td>
<td>287</td>
<td>.9503319</td>
<td>7.299117</td>
<td>-30.8334</td>
<td>20.71608</td>
</tr>
</tbody>
</table>

TABLE 1 – U.S. GDP growth ($GDP_g$) and S&P 500 growth ($SP500_g$)

$GDP_g$ has a mean of 0.78 with a standard deviation of 0.94. We find the minimum value, which is -2.60%, in the fourth quarter of 1957. The maximum value, which is 3.93%, is in the fourth quarter of 1949. $SP500_g$ has a mean of 0.95 and a standard deviation of 7.30. The maximum value, which is 20.71%, is in the first quarter of 1974. The minimum value, which is -30.83%, is found in the third quarter of 1974.

$SP500_g$ has a higher mean than $GDP_g$; on average, S&P 500 growths 0.17% more per quarter than U.S. GDP. $SP500_g$ has also a higher standard deviation than $GDP_g$, and can be graphically observed in FIGURE 5. The S&P 500 is a much more volatile time series than GDP.
RESULTS & DISCUSSION

The Autocorrelation Function (ACF)

The U.S. GDP growth is positively autocorrelated when the number of lags is small. It is possible to predict the U.S. GDP growth in the very short run, with ACF (1) being 0.37 and ACF (2) being 0.22. However, it is not persistent: the ACF falls rapidly to zero and then alternates steadily between positive and negative low values. The S&P 500 growth is positively autocorrelated with the lags 1, 17 and 26. However, the coefficients are small and extremely close to the 95% confidence intervals. We cannot prove a high autocorrelation for the S&P 500, and we assume it is not possible to predict the SP500g, based on the ACF (Fama, 1970).

FIGURE 6 – Autocorrelation U.S. GDP growth (GDPg_t) plot

FIGURE 7 – Autocorrelation S&P 500 growth (SP500g_t) plot
The Auto-Regressive model (AR)

Based on section *The Autocorrelation Function (ACF)*, we expect U.S. GDP growth to have statistically significant coefficients at the 5% level for its first 2 lags, while S&P 500 growth could have a small but statistically significant coefficient for its first lag. The values of the lag coefficients when computing an AR model of order L are in TABLE 2 & 3. For each model, the value of the log likelihood and the constant term is shown. The Akaike’s Information Criteria (AIC) and Bayesian Information Criteria (BIC) criteria are computed. The statistically significant coefficients at the 5% significance level shown are in bold and the lag order L that fits the data best according to the AIC and BIC criteria is highlighted in orange.

U.S. GDP growth has statistically significant coefficients at the 5% level for lags 1 and 2. In the AR (3) model the third lag is statistically significant, but also has an upper bound of its 95% confidence interval close to zero. AIC and BIC criteria differ in the lag order L that best fits the data. While the AR (3) model is the more preferred model under the AIC criteria, the BIC criteria prefer the AR (1) model since BIC penalizes additional parameters more so than AIC.

Considering both AIC and BIC criteria and comparing TABLE 2 with FIGURE 6, we conceptually conclude that the AR (2) model is the best auto-regressive model when we want to estimate the U.S. GDP growth based on past periods:

\[
GDP_{t}AR(2) = 0.78 \pm 0.32 \times GDP_{t-1} + 0.11 \times GDP_{t-2} + \varepsilon_t
\]

Where \(\varepsilon_t\) is the disturbance term that has a mean of zero.

S&P 500 growth has a statistically significant coefficient at the 5% level with lag 1. However, the 95% confidence interval lower bound of the first lag is 0.035, thus it cannot be effective in explaining the S&P 500 growth at \(t\). An extra parameter may improve the fit of the model, but it does not offset additional forecasting errors cause by its addition. Thus, the AR (1) model fits the data best according to the AIC and BIC criteria.

Comparing TABLE 3 with FIGURE 7 we conclude that the AR (1) model is the most effective auto-regressive model when explaining the S&P 500 growth at \(t\) considering previous periods:

\[
SP500_{t}AR(1) = 0.94 \pm 0.14 \times SP500_{t-1} + \omega_t
\]

Where \(\omega_t\) is the disturbance term that has mean zero.
We do an out-of-sample forecasting study for U.S. GDP growth and S&P 500 growth using the AR models established in the previous section:

\[
GDP_{g_t}AR(2) = 0.78 + 0.32 \times GDP_{g_{t-1}} + 0.11 \times GDP_{g_{t-2}} + \varepsilon_t
\]

\[
SP500_{g_t}AR(1) = 0.94 + 0.14 \times SP500_{g_{t-1}} + \omega_t
\]

We start with the fourth quarter of 1971, which is observation 100. We get a series of forecasts for each model from observation 100 to the last observation:

\[
GDP_{gp_t} = \text{series of forecast for model } GDP_{gt}.
\]

\[
SP500_{gp_t} = \text{series of forecast for model } SP500_{gt}.
\]

We compute the MSFE for the out-of-sample period:

\[
\text{MSFE GDP}_{g_t} AR (2) = \frac{1}{187} \sum_{t=1}^{187} (GDP_{g_t} - GDP_{gp_t})^2 = 0.5285
\]

\[
\text{MSFE SP500}_{g_t} AR (1) = \frac{1}{187} \sum_{t=1}^{187} (SP500_{g_t} - SP500_{gp_t})^2 = 58.92
\]
The GDPg, AR (2) fits the data better than SP500g, AR (1). We cannot predict SP500g based on the ACF, which is consistent with FIGURE 7. While it is convenient to include past GDP periods into the model to estimate GDPg, SP500g cannot be estimated using its past periods.

FIGURE 8 – Forecast of U.S. GDP growth (GDPgp) and real growth of the U.S. GDP (GDPg)

FIGURE 9 – Forecast of S&P 500 growth (SP500gp) and real growth of the S&P 500 (SP500g)

**The Vector Auto-Regressive model (VAR)**

We compute a VAR model of order 3 - VAR (3) model – and we get the following coefficient matrix for lag 3, BIC and AIC criteria:
Coefficient matrix of lag 3 = \[
\begin{bmatrix}
-0.0657195 & 0.0044145 \\
-0.7621104 & 0.0549437 \\
\end{bmatrix}
\]

AIC criteria = 9.28
BIC criteria = 9.46

None of the coefficients are statistically significant at the 5% significance level; therefore, the coefficient matrix of lag 3 is not statistically significant. We test the coefficient matrix of lag 2 – VAR (2) model – and we get the following matrix, BIC criteria and AIC criteria:

Coefficient matrix of lag 2 = \[
\begin{bmatrix}
0.1091681 & 0.0269633 \\
-0.9927755 & -0.032851 \\
\end{bmatrix}
\]

AIC criteria = 9.27
BIC criteria = 9.39

The coefficient matrix of lag 2 is 5% significant because three of its coefficients are significant at the 5% significance level. Furthermore, it has a better BIC and AIC criteria\(^3\), leading us to conclude the optimal order of our VAR model is 2.

The coefficient matrix of lag 2 is consistent with the auto-regressive models developed in section "The Auto-Regressive model (AR)"; the coefficient of the second lag of the GDP\(_t\) is statistically significant, but the coefficient of the second lag of SP500\(_t\) is not. However, the second lag of SP500\(_t\) is significant at the 5% level when estimating the real growth rate of U.S. GDP (GDP\(_t\)).

The statistic coefficients of the VAR (2) model are summarized in TABLE 4. The root mean squared error is smaller when modeling GDP\(_t\), suggesting a more accurate model with less errors. The R-squared is 24% for our GDP\(_t\) data and 3% for the SP500\(_t\) data. Both equations reject the joint null hypothesis that all the regression coefficients are zero at the 5% level\(^4\). There are 2 equations, a total of 10 parameters and 2 error terms:

\[
\begin{align*}
GDP_t \text{ VAR (2)} & = 0.48 + 0.22 \cdot GDP_{t-1} + 0.11 \cdot GDP_{t-2} + 0.029 \cdot SP500_{t-1} + 0.027 \cdot SP500_{t-2} + \varepsilon_{1,t} \\
SP500_t \text{ VAR (2)} & = 1.29 + 0.14 \cdot SP500_{t-1} - 0.03 \cdot SP500_{t-2} + 0.44 \cdot GDP_{t-1} - 0.99 \cdot GDP_{t-2} + \varepsilon_{2,t}
\end{align*}
\]

\(^3\) For a VAR (1) model the BIC criteria is 9.40 and the AIC criteria is 9.33; BIC and AIC criteria prefer the VAR (2) model.

\(^4\) Note: see section Appendix for statistical tables.
We use the matrix computation with 3 matrices and the disturbance matrix:

\[
\begin{bmatrix}
GDP_t \\
SP500_{t-1}
\end{bmatrix}
= \begin{bmatrix} 0.48 \\ 1.29 \end{bmatrix} + \begin{bmatrix} 0.22 & 0.029 \\
0.44 & 0.14 \end{bmatrix}
\begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix}
+ \begin{bmatrix} 0.11 \\
-0.99 \end{bmatrix}
\begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix}
+ \begin{bmatrix} \epsilon_{1,t} \\
\epsilon_{2,t} \end{bmatrix}
\]

There are two coefficients in the SP500g VAR (2) equation that are not statistically significant at the 5% level: the coefficient of the second lag of the SP500g, and the first lag coefficient of the GDPg. The model fits GDP data: all parameter coefficients are statistically significant, the standard errors are lower, and they all have better measurements of accuracy.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Lag</th>
<th>Coef.</th>
<th>Std. Err.</th>
<th>95% Cl</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDPg</td>
<td>1</td>
<td>0.22137</td>
<td>0.05774</td>
<td>[0.10819, 0.33455]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.10916</td>
<td>0.05539</td>
<td>[0.00059, 0.33455]</td>
</tr>
<tr>
<td>SP500g</td>
<td>1</td>
<td>0.02891</td>
<td>0.00681</td>
<td>[0.01556, 0.04226]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.02696</td>
<td>0.00704</td>
<td>[0.01316, 0.04076]</td>
</tr>
<tr>
<td>GDPg</td>
<td>1</td>
<td>1.29184</td>
<td>0.60773</td>
<td>[0.100702, 2.48299]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.43758</td>
<td>0.50530</td>
<td>[-0.55278, 1.42796]</td>
</tr>
<tr>
<td>SP500g</td>
<td>1</td>
<td>-0.99277</td>
<td>0.48473</td>
<td>[-1.94283, -0.04271]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.03285</td>
<td>0.06161</td>
<td>[-0.15361, 0.08791]</td>
</tr>
</tbody>
</table>

**TABLE 4 – GDPg and SP500g VAR (2) model coefficients results**

**Granger Causality**

The S&P 500 time series Granger-causes U.S. GDP. The test rejects the null hypothesis at the 1% significance level that the MSFE is the same whether we include information about the S&P 500 to predict the behavior of the U.S. GDP. Hence, when we forecast GDP time series, it is more accurate to model the GDPg, VAR (2) equation than the GDPg, AR (2) since data about the S&P 500 time series decreases the number of errors when estimating GDPg.

On the other hand, GDP time series is not helpful when forecasting the S&P 500. The Granger Causality test does not reject at the 10% significance the null hypothesis that the MSFE is the same whether we include GDP time series to predict the behavior of the S&P 500. In other words, lagged values of the U.S. GDP do not contribute to forecast the S&P 500:

\[
\begin{bmatrix}
GDP_t \\
SP500_{t-1}
\end{bmatrix}
= \begin{bmatrix} 0.48 \\ 1.29 \end{bmatrix} + \begin{bmatrix} 0.22 & 0.029 \\
0.14 & 0.14 \end{bmatrix}
\begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix}
+ \begin{bmatrix} 0.11 \\
0 \end{bmatrix}
\begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix}
+ \begin{bmatrix} \epsilon_{1,t} \\
\epsilon_{2,t} \end{bmatrix}
\]

Granger Causality reinforces the statement in section *The Vector Auto-Regressive model (VAR)* that the model fits GDP data better than SP500 data. Information about Standard and Poor’s 500
should be included when forecasting the U.S. GDP, but including data about U.S. GDP should have no impact when predicting Standard and Poor’s 500 behavior.

It seems more efficient to use the SP500g, AR (1) \(^5\) than the SP500g, VAR (2) equation when forecasting the S&P 500: it has less parameters, requires less data, and has similar predictability. However, as we demonstrated in section \textit{AR Forecasting}, the SP500g, AR (1) is not a good model to predict the S&P 500. Therefore, the model we should analyze is the GDPg, VAR (2) equation.

\textbf{Granger causality Wald tests}

<table>
<thead>
<tr>
<th>Equation</th>
<th>Excluded</th>
<th>(\text{chi}^2)</th>
<th>df</th>
<th>Prob &gt; (\text{chi}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDPg</td>
<td>SP500g</td>
<td>36.464</td>
<td>2</td>
<td>0.000</td>
</tr>
<tr>
<td>GDPg</td>
<td>ALL</td>
<td>36.464</td>
<td>2</td>
<td>0.000</td>
</tr>
<tr>
<td>SP500g</td>
<td>GDPg</td>
<td>4.2311</td>
<td>2</td>
<td>0.121</td>
</tr>
<tr>
<td>SP500g</td>
<td>ALL</td>
<td>4.2311</td>
<td>2</td>
<td>0.121</td>
</tr>
</tbody>
</table>

\textit{TABLE 5 – Granger Causality test results}

\textbf{Impulse Response Functions}

In FIGURE 10 there are the Impulse Response Functions for both time series. On the left, there is the Impulse Response Function of the SP500g\(_t\) to a shock in the GDPg\(_t\). On the right, the response variable is the GDPg\(_t\) and the impulse variable is the SP500g\(_t\).

We cannot reject the hypothesis that the S&P 500 time series is not affected when there is a shock on U.S. GDP: the 95% Confidence Interval does include 0 for all \(t\) in the left figure. However, a shock in the S&P 500 time series does have an effect of the same sign in the GDP time series: a positive shock in the S&P 500 is expected to be correlated with an increase in U.S. GDP. The reaction is short-term, leading the series with a maximum effect 1-2 quarters after its introduction and rapidly reverting to the initial level afterwards.

Whereas a shock in the U.S. GDP time series does not predict the movement of the Standard and Poor’s 500, a shock in the Standard and Poor’s 500 does positively correlate with U.S. GDP in the short-run. The conclusion is consistent with the results found in section \textit{Granger Causality}: information about Standard and Poor’s 500 is useful when forecasting the U.S. GDP, but

\(^5\) \textit{Note: the best linear prediction for SP500g\(_t+s\), given its past periods is the SP500g, AR (1) developed in section \textit{The Auto-Regressive model (AR)}: Proj(SP500g\(_{t+s}|SP500g\(_t,SP500g\(_{t-1},...\)) = 0.94 + 0.14 * SP500g\(_{t-1}\)
including data about U.S. GDP has no impact when predicting the S&P 500. The only model we should evaluate is the GDPt, VAR (2) equation.

\[
GDP_{t,AR}(2) = 0.78 + 0.32 \cdot GDP_{t-1} + 0.11 \cdot GDP_{t-2} + \epsilon_t
\]

\[
GDP_{t,VAR}(2) = 0.48 + 0.22 \cdot GDP_{t-1} + 0.11 \cdot GDP_{t-2} + 0.029 \cdot SP500_{t-1} + 0.027 \cdot SP500_{t-2} + \epsilon_{t,t}
\]

We do an out-of-sample forecasting study for the GDPt, VAR (2) model and the GDPt, AR (2) model:

We start with the fourth quarter of 1971, which is the observation number 100. We get a series of forecasts for each model from observation 100 to the last observation:

\[\text{GDP}_{gp,t} = \text{series of forecast for model GDP}_{t, AR}(2)\]

\[\text{GDP}_{gx,t} = \text{series of forecast for model GDP}_{t, VAR}(2)\]

In FIGURE 11 we observe the red line is the forecast of the U.S. GDP with the GDPgt AR (2) model, the green line is the forecast with the GDPgt VAR (2) model and the blue line is the actual growth of the U.S. GDP. We observe that the GDPgt VAR (2) seems to fit data better than GDPgt AR (2).
The mean-squared-forecast-errors (MSFE) and its square root for the out-of-sample period are computed:

$$\text{MSFE GDP}_t^{\text{AR (2)}} = \frac{1}{187} \sum_{t=1}^{187} (GDP_t - GDP_{gp_t})^2 = 0.5285;$$

$$\sqrt{\text{MSFE GDP}_t^{\text{AR (2)}}} \approx 0.73$$

$$\text{MSFE GDP}_t^{\text{VAR (2)}} = \frac{1}{187} \sum_{t=1}^{187} (GDP_t - GDP_{gx_t})^2 = 0.4782$$

$$\sqrt{\text{MSFE GDP}_t^{\text{VAR (2)}}} \approx 0.69$$

GDP$_t^{\text{VAR (2)}}$ model errors less compared to the GDP$_t^{\text{AR (2)}}$ model when forecasting the real growth of U.S. GDP. On average, the GDP$_t^{\text{AR (2)}}$ model errors 73 basis points of difference between the estimation of U.S. GDP growth and the actual U.S. GDP growth, while the GDP$_t^{\text{VAR (2)}}$ model errors 69 basis points on average. Thus, there is a difference of 4 basis points in accuracy, or a 5% improvement when we estimate using the GDP$_t^{\text{VAR (2)}}$ model.

It is convenient to include data about the S&P 500 to estimate the real growth of the U.S. GDP. Based on sections Granger Causality, Impulse Response Functions and VAR Forecasting, the model this study considers is the GDP$_t^{\text{VAR (2)}}$ equation.

$$GDP_{gt}^{\text{VAR (2)}} = 0.48 + 0.22 \cdot GDP_{gt-1} + 0.11 \cdot GDP_{gt-2} + 0.029 \cdot SP500_{gt-1} + 0.027 \cdot SP500_{gt-2} + \epsilon_{gt}$$

(0.07)  (0.06)  (0.06)  (0.01)  (0.01)
CONCLUSION

From our econometric relation analysis on estimating quarterly U.S. GDP growth, we have found:

- U.S. GDP growth is positively autocorrelated in the short-run, when the lag order is small.
- There is a positive and significant historical relation between current U.S. GDP growth and S&P 500 growth from the previous two quarters.
- The explanatory power of previous quarters’ U.S. GDP and S&P 500 growths increases with relative proximity to the current quarter and past evolution of U.S. GDP growth has a higher explanatory power than S&P 500 growth.

From our econometric relation analysis on estimating quarterly S&P 500 growth, we have found:

- S&P 500 growth is a volatile time series and cannot be estimated using past periods.

In conclusion, forecasting the S&P 500 is almost impossible, and no improvements were found with the inclusion of U.S GDP data: U.S. GDP growth is not an indicator for S&P 500 growth. However, S&P 500 growth has a predictive power over U.S. GDP time series and a shock in the S&P 500 will shortly be observed in the U.S. GDP growth: S&P 500 growth is a short-run leading indicator for U.S. GDP growth and its inclusion led to a 5% accuracy improvement in our forecasted outcomes.

The results are consistent with the Vector Error-Correction Model elaborated by McMillan (2005) and a Dynamic Factor Model (Stock and Watson, 2011) should reaffirm the conclusions. The unforeseeability of stock markets given past returns is a point far from new (Fama, 1970; Comincioli, 1996; Ritter, 2005). However, the fact that including U.S. GDP data does not improve the forecast is not so well known. Interpreting stock markets as a leading indicator is consistent with Campbell (1987), Cochrane (1991) and Levine and Zervos (1998), but additional studies could look to control different sampling periods to observe more precise cyclical tendencies. Perhaps a similar study could be conducted in the next decade to test whether there have been any diverging growth patterns since the Great Recession. All in all, further research should be done.
REFERENCES


APPENDIX

Mathematical Approach

Impulse Response Functions

Assume time series of real U.S. GDP growth are denoted GDPg and time series of the real Standard & Poor’s 500 growth are denoted SP500g. We rewrite the VAR \((p)\) model to be a function of its residuals:

\[
\begin{bmatrix}
GDP_{t} \\
SP500_{t}
\end{bmatrix} = \vartheta_0 + \varphi_{0,1} * \begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix} + \cdots + \varphi_{0,p} * \begin{bmatrix}
GDP_{t-p} \\
SP500_{t-p}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t}
\end{bmatrix}
\]

Where:

\[
\begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix} = \vartheta_1 + \varphi_{1,1} * \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \cdots + \varphi_{1,p} * \begin{bmatrix}
GDP_{t-p} \\
SP500_{t-p}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-1} \\
\varepsilon_{2,t-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} = \vartheta_2 + \varphi_{2,1} * \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \cdots + \varphi_{2,p} * \begin{bmatrix}
GDP_{t-p} \\
SP500_{t-p}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-2} \\
\varepsilon_{2,t-2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
GDP_{t-d} \\
SP500_{t-d}
\end{bmatrix} = \vartheta_d + \varphi_{d,1} * \begin{bmatrix}
GDP_{t-d} \\
SP500_{t-d}
\end{bmatrix} + \cdots + \varphi_{d,p} * \begin{bmatrix}
GDP_{t-p} \\
SP500_{t-p}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-d} \\
\varepsilon_{2,t-d}
\end{bmatrix}
\]

And:

\[
\varphi_{d,p} = \begin{bmatrix}
\beta_{p,11} & \beta_{p,12} \\
\beta_{p,21} & \beta_{p,22}
\end{bmatrix}
\]

for the VAR \((p)\) model at \(t-d\)

For a VAR \((2)\) model:

\[
\begin{bmatrix}
GDP_{t} \\
SP500_{t}
\end{bmatrix} = \vartheta_0 + \varphi_{0,1} * \begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix} + \varphi_{0,2} * \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t}
\end{bmatrix}
\]

Where:

\[
\begin{bmatrix}
GDP_{t-1} \\
SP500_{t-1}
\end{bmatrix} = \vartheta_1 + \varphi_{1,1} * \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \varphi_{1,2} * \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-1} \\
\varepsilon_{2,t-1}
\end{bmatrix}
\]
\[
\begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix}
= \vartheta_2 + \varphi_{2,1} \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \varphi_{2,2} \begin{bmatrix}
GDP_{t-4} \\
SP500_{t-4}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-2} \\
\varepsilon_{2,t-2}
\end{bmatrix}
\]

We put together all the above matrices using the substitution method:

\[
\begin{bmatrix}
GDP_t \\
SP500_t
\end{bmatrix}
= \vartheta_0 + \varphi_{0,1} \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t}
\end{bmatrix} + \varphi_{0,2} \begin{bmatrix}
\varepsilon_{1,t-1} \\
\varepsilon_{2,t-1}
\end{bmatrix} + \varphi_{1,1} \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \varphi_{1,2} \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \varphi_{0,1} \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \varphi_{1,1} \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \varphi_{1,2} \begin{bmatrix}
GDP_{t-4} \\
SP500_{t-4}
\end{bmatrix}
\]

We rearrange the terms and eliminate constant terms:\

\[
\begin{bmatrix}
GDP_t \\
SP500_t
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t}
\end{bmatrix} + \varphi_{0,1} \begin{bmatrix}
\varepsilon_{1,t-1} \\
\varepsilon_{2,t-1}
\end{bmatrix} + \varphi_{0,2} \begin{bmatrix}
\varepsilon_{1,t-2} \\
\varepsilon_{2,t-2}
\end{bmatrix} + \varphi_{1,1} \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \varphi_{0,1} \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} + \varphi_{1,1} \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \varphi_{0,2} \begin{bmatrix}
GDP_{t-4} \\
SP500_{t-4}
\end{bmatrix}
\]

If the model is stationary, the regressors coefficients are smaller than one. When \(d \to \infty\), the VAR (\(p\)) model can be approximately modeled in the following matrix equation\(^7\) using geometric series properties:

\[
\lim_{d \to \infty} \begin{bmatrix}
GDP_t \\
SP500_t
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t}
\end{bmatrix} + \varphi_0 \begin{bmatrix}
\varepsilon_{1,t-1} \\
\varepsilon_{2,t-1}
\end{bmatrix} + \varphi_1 \begin{bmatrix}
\varepsilon_{1,t-2} \\
\varepsilon_{2,t-2}
\end{bmatrix} + \ldots + \varphi_d \begin{bmatrix}
\varepsilon_{1,t-3} \\
\varepsilon_{2,t-3}
\end{bmatrix}
\]

---

\(^6\)Note: we could have substituted \( \begin{bmatrix}
GDP_{t-2} \\
SP500_{t-2}
\end{bmatrix} = \vartheta_2 + \varphi_{2,1} \begin{bmatrix}
GDP_{t-3} \\
SP500_{t-3}
\end{bmatrix} + \varphi_{2,2} \begin{bmatrix}
GDP_{t-4} \\
SP500_{t-4}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{1,t-2} \\
\varepsilon_{2,t-2}
\end{bmatrix} \)

but it would have resulted in a large useless equation.

\(^7\) Note: \( \varphi_d \neq \varphi_{d,0} \)
The Auto-Regressive model (AR)

The Auto-Regressive model (AR) has been modeled using STATA software.

```
.arima GDPg, arima(2,0,0)
(setting optimization to BHHH)
Iteration 0: log likelihood = -367.68305
Iteration 1: log likelihood = -367.68066
Iteration 2: log likelihood = -367.68059
Iteration 3: log likelihood = -367.68058

ARIMA regression
Sample: 1 - 287
Number of obs = 287
Wald ch2(2) = 59.96
Log likelihood = -367.6806
Prob > ch2 = 0.0000

| GDPg      | Coef.   | Std. Err. | z     | P>|z|  | 95% Conf. Interval |
|-----------|---------|-----------|-------|------|-------------------|
| GDPg      | _cons   | 0.7791712 | 0.0930532 | 8.37 | 0.000 | 0.5967902 - 0.9615521 |
| ARMA      |         |           |       |      |                   |
| L1.       | ar      | 0.3194235 | 0.0482898 | 6.61 | 0.000 | 0.2247771 - 0.4140698 |
| L2.       |         | 0.1137654 | 0.0490266 | 2.32 | 0.020 | 0.017675 - 0.2098558 |
| /sigma    |         | 0.8710647 | 0.0259786 | 33.53 | 0.000 | 0.8201475 - 0.9219819 |
```

Note: The test of the variance against zero is one sided, and the two-sided confidence interval is truncated at zero.

APENDIX STATA 1 – GDPg AR (2) model
. arima SP500g, arima(1,0,0)

(setting optimization to BHHH)
Iteration 0:  log likelihood =  -974.45744
Iteration 1:  log likelihood =  -974.45678
Iteration 2:  log likelihood =  -974.45656
Iteration 3:  log likelihood =  -974.45648
Iteration 4:  log likelihood =  -974.45645
(switching optimization to BFGS)
Iteration 5:  log likelihood =  -974.45644

ARIMA regression

Sample:  1 - 287
Number of obs  =  287
Wald chi2(1)   =  6.89
Log likelihood =  -974.4564
Prob > chi2    =  0.0087

<table>
<thead>
<tr>
<th></th>
<th>OPG</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coef.    Std. Err.   z   P&gt;</td>
</tr>
<tr>
<td>SP500g</td>
<td>Coef.    Std. Err.   z   P&gt;</td>
</tr>
<tr>
<td>_cons</td>
<td>0.9417013 0.5728924  1.64 0.100  -0.1811472  2.06455</td>
</tr>
<tr>
<td></td>
<td>ARMA</td>
</tr>
<tr>
<td></td>
<td>Coef.    Std. Err.   z   P&gt;</td>
</tr>
<tr>
<td></td>
<td>ar</td>
</tr>
<tr>
<td></td>
<td>0.1386055 0.0527961  2.63 0.009  0.0351271  0.2420838</td>
</tr>
<tr>
<td></td>
<td>l1</td>
</tr>
<tr>
<td></td>
<td>7.216913  0.2211912 32.63 0.000  6.783386  7.65044</td>
</tr>
</tbody>
</table>

Note: The test of the variance against zero is one sided, and the two-sided confidence interval is truncated at zero.

APENDIX STATA 2 – SP500g: AR (1) model
The Vector Auto-Regressive model (VAR)

The Vector Auto-Regressive model (VAR) has been modeled using STATA software. Note: STATA does not do an estimation for the first X quarters for the VAR (X) model because it does not have data for its lag periods. Hence, the number of observations is \((287 - X)\) instead of 287.

**Vector autoregression**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Parms</th>
<th>RMSE</th>
<th>R-sq</th>
<th>ch12</th>
<th>P&gt;ch12</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDPg</td>
<td>3</td>
<td>0.85214</td>
<td>0.1862</td>
<td>65.42633</td>
<td>0.0000</td>
</tr>
<tr>
<td>SP500g</td>
<td>3</td>
<td>7.26712</td>
<td>0.0191</td>
<td>5.582353</td>
<td>0.0613</td>
</tr>
</tbody>
</table>

|                | Coef.  | Std. Err. | z     | P>|z|  | 95% Conf. Interval |
|----------------|--------|-----------|-------|------|-------------------|
| GDPg           |        |           |       |      |                   |
| GDPg L1.       |0.3216598 |0.0539225 | 5.97 | 0.000 |0.2159737 -0.4273459 |
| SP500g L1.     |0.0310368 |0.0069944 | 4.44 | 0.000 |0.0173289 -0.0447446 |
| _cons          |0.5036807 |0.0651742 | 7.73 | 0.000 |0.3759416 -0.6314198 |
| SP500g         |        |           |       |      |                   |
| GDPg L1.       |0.0199935 |0.4598557 | 0.04 | 0.965 |-0.8813071 0.921294 |
| SP500g L1.     |0.1386687 |0.0596453 | 2.32 | 0.020 |0.0217662 0.2555714 |
| _cons          |0.7967415 |0.5558115 | 1.43 | 0.152 |-0.2926289 1.886112 |

**APENDIX STATA 3 – GDPg and SP500g VAR (1) model**
Vector autoregression

Sample: 3 - 287  
Number of obs = 285

Log likelihood = -1310.5  
AIC = 9.266667

FPE = 36.26742  
HQIC = 9.318042

Det(Sigma_ml) = 33.80934  
SBIC = 9.394824

<table>
<thead>
<tr>
<th>Equation</th>
<th>Parms</th>
<th>RMSE</th>
<th>R-sq</th>
<th>ch12</th>
<th>P&gt;ch12</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDPg</td>
<td>5</td>
<td>.827775</td>
<td>0.2372</td>
<td>88.62378</td>
<td>0.0000</td>
</tr>
<tr>
<td>SP500g</td>
<td>5</td>
<td>7.24346</td>
<td>0.0349</td>
<td>10.2983</td>
<td>0.0357</td>
</tr>
</tbody>
</table>

| GDPg      | Coef. | Std. Err. | z     | P>|z|   | [95% Conf. Interval] |
|-----------|-------|-----------|------|------|----------------------|
| GDPg      |       |           |      |      |                      |
| L1.       | .2213769 | .0577455 | 3.83 | 0.000 | .1681979 -- .3345559 |
| L2.       | .1091681 | .0553946 | 1.97 | 0.049 | .0005967 -- .2177396 |
| SP500g    |       |           |      |      |                      |
| L1.       | .0289191 | .0068113 | 4.25 | 0.000 | .0155692 -- .042269  |
| L2.       | .0369633 | .0070412 | 3.83 | 0.000 | .0131528 -- .0407638 |
| _cons     | .4750744 | .0694516 | 6.84 | 0.000 | .3389516 -- .6111971 |

| SP500g    | Coef. | Std. Err. | z     | P>|z|   | [95% Conf. Interval] |
|-----------|-------|-----------|------|------|----------------------|
| GDPg      |       |           |      |      |                      |
| L1.       | .4375866 | .5053028 | 0.87 | 0.386 | -.5527886 -- 1.427962 |
| L2.       | -.9927755 | .4847319 | -2.05 | 0.041 | -.1942833 -- -.8427184 |
| SP500g    |       |           |      |      |                      |
| L1.       | .137901 | .0596023 | 2.31 | 0.021 | .0210827 -- .2547194  |
| L2.       | -.032851 | .0616141 | -0.53 | 0.594 | -.1536124 -- .0879104 |
| _cons     | 1.291847 | .5077381 | 2.13 | 0.034 | .8007022 -- 2.482992  |

APENDIX STATA 4 – GDPg and SP500g VAR (2) model
### APENDIX STATA 5 – GDPg and SP500g VAR (3) model

**Sample:** 4 – 287  
**Number of obs** = 284  
**Log likelihood** = -1303.474  
**AIC** = 9.277983  
**FPE** = 36.68065  
**HOIC** = 9.350101  
**Det(Sigma_ml)** = 33.23614  
**SBIC** = 9.457862

<table>
<thead>
<tr>
<th>Equation</th>
<th>Parms</th>
<th>RMSE</th>
<th>R-sq</th>
<th>chi2</th>
<th>P&gt;chi2</th>
</tr>
</thead>
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<tr>
<td>GDPg</td>
<td>7</td>
<td>0.825463</td>
<td>0.2460</td>
<td>92.63671</td>
<td>0.0000</td>
</tr>
<tr>
<td>SP500g</td>
<td>7</td>
<td>7.24104</td>
<td>0.0453</td>
<td>13.48413</td>
<td>0.0360</td>
</tr>
</tbody>
</table>

| GDPg        | Coef. | Std. Err. | z     | P>|z|  | 95% Conf. Interval     |
|-------------|-------|-----------|-------|------|------------------------|
| GDPg        |       |           |       |      |                        |
| L1.         | 0.2264524 | 0.0596999 | 3.79  | 0.000 | 0.1094427 - 0.3434622 |
| L2.         | 0.1275929 | 0.059143  | 2.16  | 0.030 | 0.01263 0.2429728 |
| L3.         | -0.0657195 | 0.0560291 | -1.17 | 0.241 | -0.1755345 0.0440955 |
| SP500g      |       |           |       |      |                        |
| L1.         | 0.0283642 | 0.068621  | 4.13  | 0.000 | 0.0149146 0.0418137 |
| L2.         | 0.0260525 | 0.0670258 | 3.71  | 0.000 | 0.0122623 0.0398228 |
| L3.         | 0.0044145 | 0.071991  | 0.61  | 0.540 | -0.0996954 0.0185244 |
| _cons       | 0.500831  | 0.0747314 | 6.70  | 0.000 | 0.3543602 0.6473018 |

| SP500g      |       |           |       |      |                        |
|-------------|-------|-----------|-------|------|                        |
| GDPg        |       |           |       |      |                        |
| L1.         | 0.4383074 | 0.5230597 | 0.84  | 0.402 | -0.956709 1.463486 |
| L2.         | -0.8429435 | 0.5161763 | -1.63 | 0.102 | -1.85463 0.1687434 |
| L3.         | -0.7621104 | 0.4988976 | -1.55 | 0.121 | -1.724252 0.2006314 |
| SP500g      |       |           |       |      |                        |
| L1.         | 0.1278861 | 0.0601225 | 2.13  | 0.033 | 0.0100481 0.245724 |
| L2.         | -0.0399726 | 0.061556  | -0.65 | 0.516 | -0.1606203 0.080675 |
| L3.         | 0.0549437 | 0.0630745 | 0.87  | 0.384 | -0.06668 0.1785574 |
| _cons       | 1.742742  | 0.6547574 | 2.66  | 0.008 | 0.459441 3.026043 |