Distribution of the Number of Users per Base Station in Cellular Networks

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***Abstract*—We consider the number of users associating with each base station in a cellular network. Extending and unifying the characterizations for certain settings available in the literature, we derive a result that is asymptotic in the strength of the shadowing, yet otherwise universally valid: it holds for every network geometry and shadowing distribution. We then illustrate how this result provides excellent representations in various classes of networks and with realistic shadowing strengths, evidencing broad applicability.**

***Index Terms*—Cellular networks, user association, number of users, shadowing, stochastic geometry, multiuser MIMO, Poisson point process, lattice networks, PPP networks**

I. Motivation

The analysis of cellular networks via Poisson point process (PPP) modeling of the base station (BS) locations is a welcome complement, and sometimes even an outright alternative, to the Monte-Carlo simulations that had long dominated system-level performance evaluations [1]–[3]. Such analysis, seemingly fitting only for ad hoc networks, happens to be highly relevant to cellular networks by virtue of the following result: regardless of the BS locations, provided only that they are agnostic to the radio propagation, the distribution of powers received at any user converges (asymptotically in the strength of the shadowing) to what would be received from a PPP field of BSs [4]–[6]. This convergence, moreover, is very evident for practical strengths of the shadowing. In hexagonal lattice networks, for instance, PPP-based analyses are highly representative for shadowing standard deviations on the order of 10 dB [6]–[8], well in line with the typical values encountered in macrocellular deployments.

An issue that arises in the analysis of cellular networks is the modeling of the number of users associating with each BS. While hidden if each BS is assumed to communicate with a single user per signaling resource, this issue becomes material once that assumption is removed, say in the face of multiantenna BSs capable of implementing multiuser multiple-input multiple-output transmission [9]. Even in a lattice network whose cells are of equal size, the shadowing and the stochastic nature of the user locations would induce disparities in the number of users associating with different BSs, and such disparities are bound to increase in irregular networks.

This letter addresses the stochastic modeling of the number of users per BS. An approximate characterization available in the literature is discussed, and a new asymptotic characterization is provided and tested.

II. Modeling Features

Our focus is on cellular networks where users associate with the BS from which they enjoy the strongest large-scale channel gain. Let us next describe the essential modeling features of the networks to which the considerations in the sequel apply.

A. Geometries

In terms of the positions of BSs and users, virtually every cellular scenario of relevance is encompassed. The BS locations may conform to any stationary and ergodic point process $\Phi_{\mathsf{b}} \subset \mathbb{R}^2$ of density λ_{b}, or any realization thereof, say a lattice network. This implies that the density of BSs within any region converges to $\lambda_{\mathsf{b}} > 0$ as this region's area grows [6]. Meanwhile, the user positions $\Phi_{\mathsf{u}} \subset \mathbb{R}^2$ may belong to any independent point process of density λ_{u} that is also stationary and ergodic.

Without loss of generality, a specific BS is set at the origin. In the random case, we condition on a BS to be located at the origin; under expectation over Φ_{b}, this becomes the typical BS. In the deterministic case, we pick an arbitrary BS and translate the coordinate system so that this BS is located at the origin. In both cases, we label this BS at the origin as the 0th BS. Denoting by K the number of users associating with such 0th BS, our purpose is to inspect the distribution of K under expectations over Φ_{u} and Φ_{b}.

B. Large-scale Gains

The large-scale channel gains include path loss with exponent $\eta > 2$ and shadowing that is IID across links. Particularly, between the ℓth BS and the kth user served by the 0th BS, the large-scale gain is

$$G_{\ell,(k)} = \frac{L_{\mathsf{ref}}}{r_{\ell,(k)}^{\eta}} \chi_{\ell,(k)} \quad \ell \in \mathbb{N}_0,\, k \in \{0, \ldots, K-1\}, \tag{1}$$

with L_{ref} the path loss intercept at a unit distance, $r_{\ell,(k)}$ the link distance, and $\chi_{\ell,(k)}$ the shadowing coefficient having standard deviation σ_{dB}. The shadowing can be arbitrarily distributed, with the only mild restriction that $\mathbb{E}\big[\chi^{2/\eta}\big] < \infty$ to guarantee the asymptotic behavior advanced in Section I.

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III. Distribution of K

A. Shadowless Networks

Suppose that there is no shadowing ($\sigma_{\rm dB} = 0$), such that the users ing with a BS are strictly the ones positioned within its Voronoi cell. Given the area of the Voronoi cell, K is a random variable with mean $\lambda_{\sf u}$ times that area [10]. From $\Phi_{\sf u}$ and the distribution of the cell area, one can then determine how K is distributed. For instance, if $\Phi_{\sf u}$ is PPP and the BSs conform to a regular lattice, then K is Poisson-distributed with mean $\bar{K} = \lambda_{\sf u}/\lambda_{\sf b}$ [10]. In turn, if both $\Phi_{\sf u}$ and $\Phi_{\sf b}$ are PPPs, then the probability mass function (PMF) of K is tightly approximated by [11], [12]

$$f_K(\mathsf{k}) \approx \frac{\Gamma(\mathsf{k}+\mathsf{c})}{\Gamma(\mathsf{k}+1)\,\Gamma(\mathsf{c})} \frac{\bar{K}^{\mathsf{k}}\,\mathsf{c}^{\mathsf{c}}}{(\mathsf{c}+\bar{K})^{\mathsf{k}+\mathsf{c}}} \qquad \mathsf{k} \in \mathbb{N}_0 \tag{2}$$

where $\mathsf{c} = 3.575$ and $\bar{K} = \lambda_{\sf u}/\lambda_{\sf b}$ is again the mean.

The characterization in (2) is appropriate to represent deployments where the BS locations are truly irregular and shadowing is minimal. However, as developed next, care must be exercised in other situations. In particular, (2) turns out to be inadequate for situations where the PPP model for $\Phi_{\sf b}$ intends to abstract the impact of shadowing on the propagation rather than the irregularity of the actual BS locations themselves.

B. Impact of Shadowing

With shadowing, users need not associate with the BS in whose Voronoi cell they are located, and hence the premise underpinning the foregoing subsection ceases to hold. Users may now associate with more distant BSs, increasingly so as $\sigma_{\rm dB}$ grows large, and characterizing the exact distribution of K in broad generality appears unwieldy.

To bypass this obstacle, we proceed to establish the distribution of K for $\sigma_{\rm dB} \to \infty$ and then test the validity of the result for values of interest for $\sigma_{\rm dB}$.

Lemma 1. *For $\sigma_{\rm dB} \to \infty$, the distribution of K becomes Poisson with mean $\bar{K} = \lambda_{\sf u}/\lambda_{\sf b}$, i.e.,*

$$f_K(\mathsf{k}) = \frac{\bar{K}^{\mathsf{k}}\, e^{-\bar{K}}}{\mathsf{k}!} \qquad \mathsf{k} \in \mathbb{N}_0. \tag{3}$$

Proof. Consider a region of finite area $\mathcal{A}$ on $\mathbb{R}^2$ having $\mathcal{A}\lambda_{\sf b}$ BSs and $\mathcal{A}\lambda_{\sf u}$ users placed arbitrarily. As $\sigma_{\rm dB} \to \infty$, the number of users associated with each BS becomes binomially distributed,

$$K \sim B\left(\mathcal{A}\lambda_{\sf u}, \frac{1}{\mathcal{A}\lambda_{\sf b}}\right), \tag{4}$$

because in the limit each user has equal probability, $\frac{1}{\mathcal{A}\lambda_{\sf b}}$, to associate with any of the $\mathcal{A}\lambda_{\sf b}$ BSs. Letting $\mathcal{A} \to \infty$ while keeping $\lambda_{\sf u}/\lambda_{\sf b}$ constant, the binomial distribution converges to the Poisson distribution with mean $\lambda_{\sf u}/\lambda_{\sf b}$ [10]. ∎

We hasten to emphasize that the foregoing result, while asymptotic, is general in terms of the network geometry: it holds for arbitrary placements of BSs and users because, as the shadowing strengthens without bound, it comes to dominate over the path loss and, ultimately, all BSs become equally likely to be the one that a user associates with.

IV. Applicability

Next, we examine the applicability of Lemma 1 to relevant classes of settings with $\sigma_{\rm dB}$ having values of practical interest. In every case, the user locations $\Phi_{\sf u}$ conform to a homogeneous PPP and the shadowing is log-normal. In the histograms obtained through Monte-Carlo, the number of network snapshots is set to ensure a 95% confidence interval of $\pm 0.07\%$ (absolute value) in the corresponding CDFs.

A. Deterministic Lattice Networks

Let the BSs be located on a lattice, in which case, as indicated earlier, K is Poisson-distributed in the absence of shadowing ($\sigma_{\rm dB} = 0$). Since each user associates with one and only one BS, the number of users-BS associations needs to be conserved regardless of $\sigma_{\rm dB}$. Also, the cells are of equal size. Intuitively then, by sheer symmetry, the distribution of K must remain unchanged for $\sigma_{\rm dB} > 0$ because the probability that the 0th BS loses the association of some number of users due to unfavorable shadowing necessarily equals the probability of gaining the same number of users from other cells due to favorable shadowing.

Indeed, for every $\sigma_{\rm dB}$, the users that stay with the 0th BS form an independent thinning of the users associating with that BS for $\sigma_{\rm dB} = 0$, so their number is always Poisson. Similarly, the users newly associating with the 0th BS form an independent thinning of the other users and thus that number is also Poisson. The sum of two independent Poisson quantities is itself Poisson, and the mean must stay constant by symmetry.

For this setting, therefore, the applicability of Lemma 1 extends to every value of $\sigma_{\rm dB}$. No matter the shadow fading, K is Poisson-distributed.

B. Deterministic Double-Lattice Networks

Consider now the network defined by

$$\Phi_{\sf b} = \mathbb{Z}^2 \cup (2\mathbb{Z})^2 + (1/2, 1/2) \tag{5}$$

and depicted in the inset of Fig. 2. Amounting to a superposition of two lattices, this network features two distinct cell sizes with an area ratio of $7/4$. While, without shadowing, each cell size maps to a distinct Poisson distribution for K, for $\sigma_{\rm dB} \to \infty$ all cells must abide by a common Poisson distribution as per Lemma 1.

Example 1. For $\eta = 4$ and $\bar{K} = 10$, with the PPP of users realized over the network described by (5), histograms of K for the two cell sizes and different values of $\sigma_{\rm dB}$ are plotted in Fig. 1. Also shown is a Poisson PMF with mean $\bar{K}$, which is the limiting distribution for both cell sizes.

Example 1 illustrates the rapid transition to the result spelled by Lemma 1, and thus the broad scope of validity thereof. This observation is bolstered by Fig. 2, where the convergence is demonstrated in terms of the variance of K. For $\sigma_{\rm dB} = 0$ dB, such variance equals 10.94 and 6.25, respectively for large and small cells, with the ratio of these values equaling that of the areas, $7/4$. But, already for $\sigma_{\rm dB} = 10$–14 dB, the variance has closely approached the limiting value of 10 for both cell sizes.

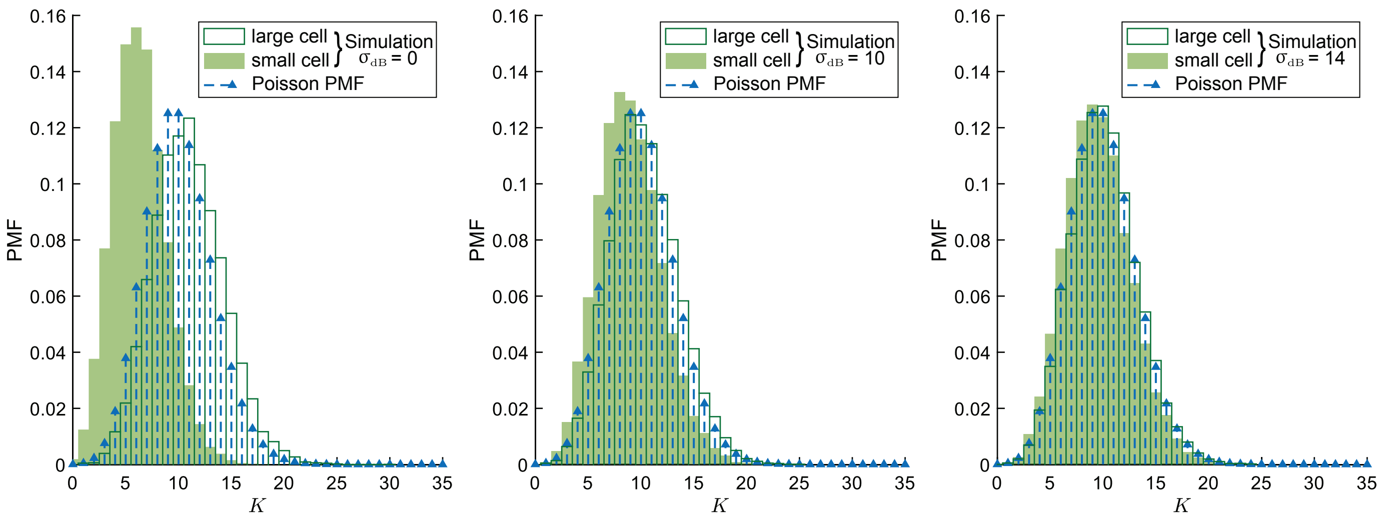

Fig. 1: In solid, histograms of K in a double-lattice network with $\eta = 4$ and $\sigma_{\text{dB}} = 0$, 10 and 14 dB (in shaded and in clear for the small and large cells, respectively); users are PPP distributed with $\bar{K} = 10$. In dashed, a Poisson PMF with mean $\bar{K}$.

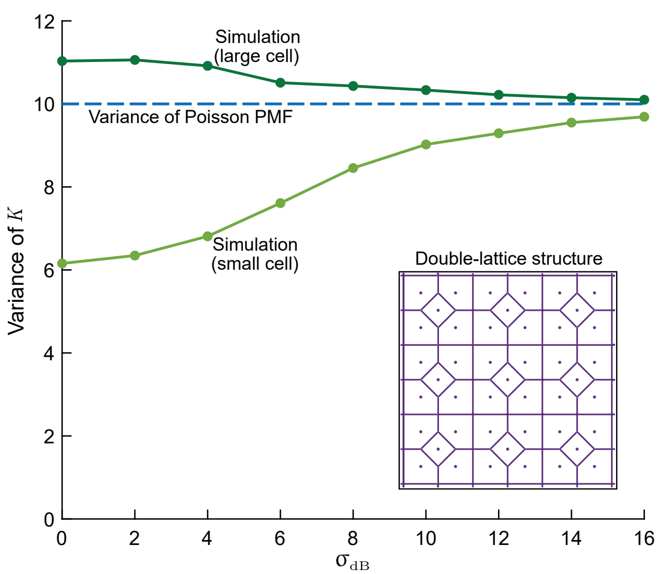

Fig. 2: In solid, variance of K as a function of σ_{dB} in a double-lattice network with $\eta = 4$; users are PPP distributed with $\bar{K} = 10$. In dashed, variance of a Poisson PMF with mean $\bar{K}$.

C. PPP Networks

Suppose now that the BS locations conform themselves to a PPP, specifically $\Phi_{\text{b}} = \Phi \cup \{o\}$ where $\Phi \subset \mathbb{R}^2$ is a homogeneous PPP and o denotes the origin [13]. Then, by Slivnyak's theorem, the central BS becomes the typical BS under expectation over Φ_{b}. In the example that follows, BSs (1000 of them on average) are randomly placed around the central one and the number of users associating with that central one is counted over Monte-Carlo snapshots.

Example 2. For $\eta = 4$ and $\bar{K} = 10$, with the PPP of users realized over a PPP network of BSs, Fig. 3 contrasts the histogram of K for different values of σ_{dB} against (2) and against a Poisson PMF with mean $\bar{K}$, corresponding respectively to the distribution of K for $\sigma_{\text{dB}} = 0$ and $\sigma_{\text{dB}} \to \infty$.

Example 2 again illustrates how the distribution of K evolves with σ_{dB}. For $\sigma_{\text{dB}} = 0$, it is very well approximated by (2), but, as the shadowing intensifies, it quickly morphs into a Poisson distribution as per Lemma 1. For $\sigma_{\text{dB}} = 10$ dB, and decidedly for $\sigma_{\text{dB}} = 14$ dB, the Poisson distribution is already an excellent match.

A complementary perspective on the evolution from (2) to the Poisson distribution within Example 2 is provided in Fig. 4, which depicts the variance of K as a function of σ_{dB}. The convergence of that variance to the variance of the Poisson PMF is almost complete for $\sigma_{\text{dB}} = 10$–14 dB. Notice the slight crossover in the vicinity of $\sigma_{\text{dB}} = 0$ dB, which serves as a measure of the (high) accuracy of (2) in this limit.

Altogether then, as in the lattice and double-lattice cases, Lemma 1 is seen to apply to practical values of σ_{dB} also in PPP networks. Moreover, all these network types are extreme cases. For randomly perturbed lattices, which have been shown to tightly fit empirical data from cellular operators [14], this conclusion is reinforced even further.

V. Discussion

Recapitulating, we can conclude the following in terms of how to stochastically model the number of users associating with each BS in a cellular network.

- For highly irregular networks subject to minimal shadowing, say certain indoor microcellular systems, the distribution in (2) represents an appropriate and precise approximation.
- For arbitrary networks subject to moderate or strong shadowing, say macrocellular systems, the Poisson distribution is the pertinent model. In the special case that the network conforms to a lattice, this is the case even if the shadowing is weak or nonexistent.

Since, as the shadowing intensifies, the number of users per BS in any network progressively behaves as in networks with equal-size cells (where it is always Poisson-distributed), we can further affirm that the shadowing acts as an equalizer of the effective cell areas. This phenomenon, whereby the

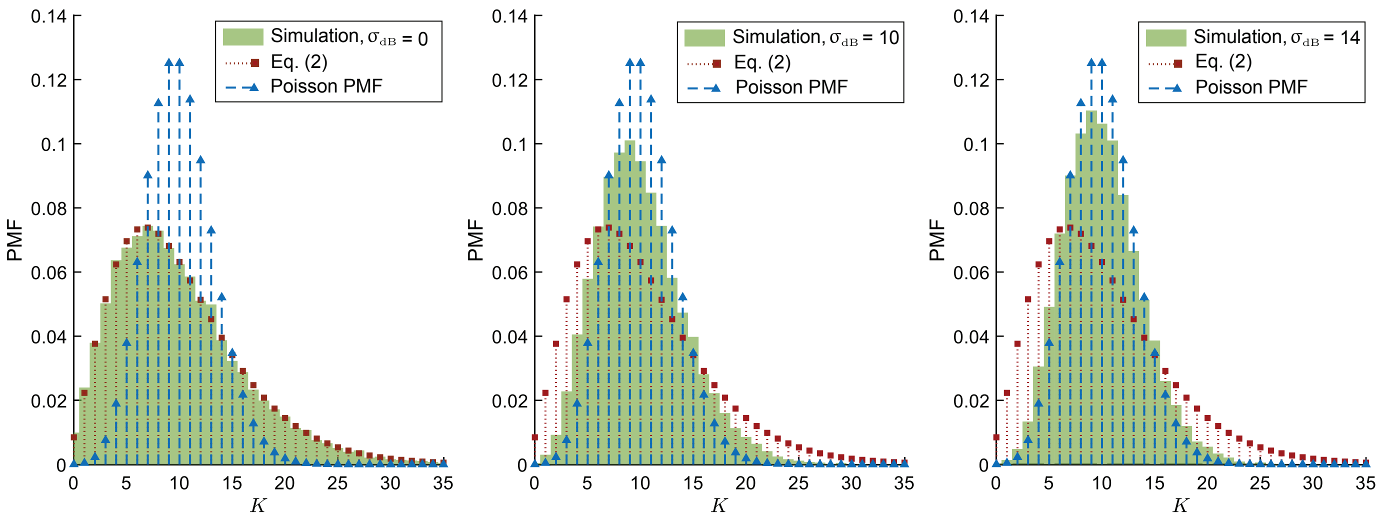

Fig. 3: In solid, histograms of K in a PPP network with $\eta = 4$ and $\sigma_{\mathrm{dB}} = 0$, 10 and 14 dB; users are PPP distributed with $\bar{K} = 10$. In dotted, the PMF in (2). In dashed, a Poisson PMF with mean $\bar{K}$.

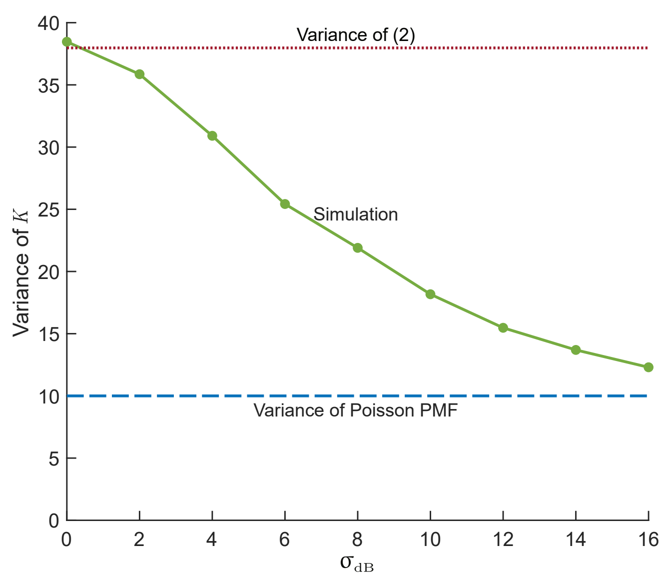

Fig. 4: In solid, variance of K as a function of σ_{dB} in a PPP network with $\eta = 4$; users are PPP distributed with $\bar{K} = 10$. In dotted, variance of the PMF in (2). In dashed, variance of a Poisson PMF with mean $\bar{K}$.

number of users per BS reduces its variance and becomes more predictable, is beneficial in terms of resource provisioning. In this respect, shadowing turns out to be operationally beneficial.

We hasten to recall that all the foregoing observations rely on the premise that the BS locations are agnostic to the radio propagation, dominated by deployment opportunities and restrictions associated with terrain, permits, infrastructure, power supply, and backhaul. At the other extreme in terms of planning we would have networks whose BS locations are optimized for coverage and service, with none of those restrictions. Such networks are best modeled by regular lattices with no shadowing [15], and again the number of users per BS would then be Poisson-distributed.

The broad suitability of a model as simple as the Poisson distribution is a welcome finding, especially if a further layer of modeling is to be overlaid so as to incorporate admission control and user selection. Such further modeling, beyond the scope of this letter, is an interesting follow-up problem.

References

[1] J. G. Andrews, F. Baccelli, and R. K. Ganti, "A tractable approach to coverage and rate in cellular networks," *IEEE Trans. Commun.*, vol. 59, no. 11, pp. 3122–3134, Nov. 2011.

[2] H. ElSawy, E. Hossain, and M. Haenggi, "Stochastic geometry for modeling, analysis, and design of multi-tier and cognitive cellular wireless networks: A survey," *IEEE Commun. Surveys & Tutorials*, vol. 15, no. 3, pp. 996–1019, Third Quarter 2013.

[3] B. Błaszczyszyn, M. Haenggi, P. Keeler, and S. Mukherjee, *Stochastic geometry analysis of cellular networks*. Cambridge University Press, 2018.

[4] H. P. Keeler, N. Ross, and A. Xia, "When do wireless network signals appear Poisson?" *Bernoulli*, vol. 24, no. 3, pp. 1973–1994, Aug. 2018.

[5] N. Ross and D. Schuhmacher, "Wireless network signals with moderately correlated shadowing still appear Poisson," *IEEE Trans. Inform. Theory*, vol. 63, no. 2, pp. 1177–1198, Feb. 2017.

[6] B. Błaszczyszyn, M. K. Karray, and H. P. Keeler, "Wireless networks appear Poissonian due to strong shadowing," *IEEE Trans. Wireless Commun.*, vol. 14, no. 8, pp. 4379–4390, Aug. 2015.

[7] G. George, R. K. Mungara, A. Lozano, and M. Haenggi, "Ergodic spectral efficiency in MIMO cellular networks," *IEEE Trans. Wireless Commun.*, vol. 16, no. 5, pp. 2835–2849, May 2017.

[8] G. George, A. Lozano, and M. Haenggi, "Massive MIMO forward link analysis for cellular networks," *submitted to IEEE Trans. Wireless Commun.*, 2018.

[9] R. W. Heath and A. Lozano, *Foundations of MIMO Communication*. Cambridge University Press, 2018.

[10] M. Haenggi, *Stochastic Geometry for Wireless Networks*. Cambridge, UK: Cambridge University Press, 2012.

[11] H. ElSawy, A. Sultan-Salem, M. S. Alouini, and M. Z. Win, "Modeling and analysis of cellular networks using stochastic geometry: A tutorial," *IEEE Communications Surveys Tutorials*, vol. 19, no. 1, pp. 167–203, Firstquarter 2017.

[12] H. ElSawy and E. Hossain, "Two-tier HetNets with cognitive femtocells: Downlink performance modeling and analysis in a multichannel environment," *IEEE Trans. Mobile Computing*, vol. 13, no. 3, pp. 649–663, Mar. 2014.

[13] Y. Wang, M. Haenggi, and Z. Tan, "The meta distribution of the SIR for cellular networks with power control," *IEEE Trans. Commun.*, vol. 66, no. 4, pp. 1745–1757, Apr. 2018.

[14] A. Guo and M. Haenggi, "Spatial stochastic models and metrics for the structure of base stations in cellular networks," *IEEE Trans. Wireless Commun.*, vol. 12, no. 11, pp. 5800–5812, Nov. 2013.

[15] ——, "Joint spatial and propagation models for cellular networks," in *Proc. IEEE Global Telecommun. Conf.*, Dec. 2015.