Retractions onto series-parallel posets

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Abstract
The poset retraction problem for a poset \( P \) is whether a given poset \( Q \) containing \( P \) as a subposet admits a retraction onto \( P \), that is, whether there is a homomorphism from \( Q \) onto \( P \) which fixes every element of \( P \). We study this problem for finite series-parallel posets \( P \). We present equivalent combinatorial, algebraic, and topological characterisations of posets for which the problem is tractable, and, for such a poset \( P \), we describe posets admitting a retraction onto \( P \).

1 Introduction
Throughout the paper, we consider only finite posets. For posets \( P \), \( Q \), etc., we shall denote their universes by \( P \), \( Q \), etc., and their partial orders by \( \leq_P \), \( \leq_Q \), etc., omitting the superscript if the order is clear from the context. A mapping \( h \) from poset \( Q \) to poset \( P \) is called monotone, or a homomorphism if, for any \( x \leq y \) in \( Q \), we have \( h(x) \leq h(y) \) in \( P \). In this case we write \( h : Q \rightarrow P \). If, in addition, \( P \) is a subposet of \( Q \) and \( h(x) = x \) for every \( x \in P \) then \( h \) is called a retraction from \( Q \) onto \( P \).

The poset retraction problem for a poset \( P \), denoted \( \text{PoRet}(P) \), is whether a given poset \( Q \) containing \( P \) as a subposet admits a retraction onto \( P \). The notion of retraction plays an important role in combinatorial theory (see, e.g., [3, 6, 7, 15]). The poset retraction problem is also of interest in theoretical computer science, as the following examples show. It can be considered a natural order-theory analogue of the much-studied graph \( H \)-coloring problem [5], or a special case of the constraint satisfaction problem (CSP), which is the problem of deciding whether there exists a homomorphism from a given relational structure to a fixed relational structure [4]. The CSP attracts much attention in artificial intelligence, combinatorics, and

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complexity theory, and it is known that a CSP for any fixed structure can be encoded as a
poset retraction problem [4]. The poset retraction problem has also been studied in connection
with type reconstruction [1, 11, 16].

In this paper, we consider the poset retraction problem in the case of series-parallel posets.
This is an interesting class of posets from both the mathematical [17, 21] and computer sci-
ence [12, 13, 14] point of view. Recall that a linear sum of two (disjoint) posets $P_1$ and $P_2$ is a
poset $P_1 + P_2$ with universe $P_1 \cup P_2$ and partial order $\leq P_1 \cup \leq P_2 \cup \{(p_1, p_2) \mid p_1 \in P_1, p_2 \in P_2\}$.

**Definition 1** A poset is called series-parallel if it can be constructed from singletons by using
disjoint union and linear sum.

Let $k$ denote a $k$-antichain (that is, disjoint union of $k$ singletons). A *4-crown* is a poset
isomorphic to $2 + 2$. The *N*-poset can be described as $2 + 2$ with one comparability missing
(its Hasse diagram looks like the letter “N”). Series-parallel posets can be characterized as
$N$-free posets, that is, posets not containing the $N$-poset as an induced subposet [20].

## 2 Definitions and Results

In this section we give all necessary definitions and state our results.

**Definition 2** A poset $P$ is said to admit an operation $f : P^n \to P$ if, for any $a, b \in P^n$ such
that $a(i) \leq b(i)$ for $i = 1, \ldots, n$, we have $f(a) \leq f(b)$. In this case we also say that $f$ is a
polymorphism of $P$.

**Definition 3** An $n$-ary operation $f$ on $P$ is called idempotent if $f(x, \ldots, x) = x$ for all
$x \in P$.

It is said to be Taylor if, in addition, it satisfies $n(\geq 2)$ identities of the form

$$f(x_{i1}, \ldots, x_{in}) = f(y_{i1}, \ldots, y_{in}), \quad i = 1, \ldots, n$$

where $x_{ij}, y_{ij} \in \{x, y\}$ for all $i, j$ and $x_{ii} \neq y_{ii}$ for $i = 1, \ldots, n$.

An $n$-ary idempotent operation $f$ is said to be totally symmetric idempotent (or TSI) if

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n),$$

for all sets of variables such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$.

Posets admitting Taylor and TSI operations were studied in [9, 10, 19]. Under the name
‘set functions’, TSI operations were applied in the study of constraint satisfaction problems [2].

**Definition 4** A subset $X$ of a poset $P$ is called an idempotent subalgebra of $P$ if there is a
triple $(Q, q_0, g)$ where $Q$ is a poset, $q_0 \in Q$, and $g$ is a partial map from $Q$ to $P$ with domain
$Y$ with the following property:

$$X = \{h(q_0) \mid h : Q \to P \text{ and } h|_Y = g\}.$$
Equivalently, $X$ is definable by a first-order formula $\phi$ with one free variable such that $\phi$ contains only conjunction, existential quantification, the order predicate (of $P$), and constants (interpreted as elements of $P$).

The name “idempotent subalgebra” comes from another equivalent definition which goes as follows. A subset of $X$ of a poset $P$ is called an idempotent subalgebra of $P$ if it is preserved by all idempotent polymorphisms of $P$, i.e., $f(x_1, \ldots, x_n) \in X$ for all $n$-ary idempotent polymorphisms of $P$ and $x_1, \ldots, x_n \in X, n \geq 1$.

Of course, an idempotent subalgebra of $P$ can be considered as a poset, with ordering induced by $P$.

For basic definitions in algebraic topology, we refer to [18]. Recall that to each finite poset is associated naturally a simplicial complex: the simplices are the totally ordered subsets of the poset (see [10] for details.) If $P$ is a connected poset, we shall denote the fundamental group of the associated complex by $\pi(P)$; we shall say that $P$ is simply connected if this group is trivial. It is well known that $\pi(2 + 2)$ is isomorphic to $\mathbb{Z}$ (see, e.g., [10]), so the 4-crown is not simply connected. It is also well-known (and easy to show) that if a poset $P$ is a retract of a poset $Q$ then the group $\pi(P)$ is a retract of $\pi(Q)$.

Denote by $X$ the class of all series-parallel posets $P$ with the following property: if \{a, a', b, b'\} is a 4-crown in $P$, a and a' being the bottom elements, then at least one of the following conditions holds:

1. there is $e \in P$ such that $a, a', e, b, b'$ form a subposet in $P$ isomorphic to $2 + 1 + 2$;
2. $\inf_P(a, a')$ exists;
3. $\sup_P(b, b')$ exists.

Recall that $\inf_P$ and $\sup_P$ denote the greatest common lower bound and the least common upper bound in $P$, respectively. Usually we write simply $\inf$ and $\sup$, as $P$ is clear from the context, and we also call them meet and join, respectively.

Fix a poset $P \in X$. If $P$ is a subposet of $Q$ and $A, B$ are two antichains in $P$, let $Q_{A,B}$ denote the set of all elements $q \in Q$ such that both $q \leq a$ for every $a \in A$ and $q \geq b$ for every $b \in B$. One of $A, B$ may be empty, and in this case the corresponding part of the condition disappears.

In order to describe posets $Q$ that admit a retraction onto $P$, we need to introduce the following two conditions on $Q$:

(R1) For every pair $(A, B)$ of antichains in $P$, $Q_{A,B}$ is empty whenever $P_{A,B}$ is empty.
(R2) For every non-empty and non-connected $H \subseteq P$ that is of the form $H = P_{A',B'}$, and for every $p_1, p_2 \in H$ that belong to different connected components of $H$, there is no path (of comparable elements) in $Q$ that connects $p_1$ and $p_2$ and such that every element in this path belongs to some $Q_{A,B}$ with $P_{A,B} \subseteq H$.

**Theorem 1** Let $P \in X$ be connected. Then a poset $Q$ containing $P$ has a retraction onto $P$ if and only if $Q$ satisfies conditions (R1) and (R2).

**Theorem 2** Let $P$ be a connected series-parallel poset. Then the following conditions are equivalent:
1. \( P \in X \);
2. \( P \) does not retract onto any of \( 1 + 2 + 2 + 2, 2 + 2 + 2 + 1, 1 + 2 + 2 + 2 + 1, 2 + 2, \) and \( 2 + 2 + 2 \);
3. \( P \) has a Taylor polymorphism;
4. \( P \) has a TSI polymorphism of every arity \( k \geq 2 \);
5. every connected idempotent subalgebra of \( P \) is simply connected;
6. for every connected idempotent subalgebra \( Q \) of \( P \), \( \pi(Q) \) does not retract onto \( \mathbb{Z} \).

**Theorem 3** Let \( P \) be a series-parallel poset. If every connected component of \( P \) satisfies any of the conditions of Theorem 2 then \( \text{PoRet}(P) \) is tractable. Otherwise it is \( \text{NP}-\text{complete} \).

### 3 Proof of Theorem 1

For a poset \( P \), denote by \( \overline{w}(P) \) and \( w(P) \) the biggest and the smallest, respectively, elements in \( P \) that are comparable with each element in \( P \).

**Lemma 1** Let \( P \in X \) be connected. Then \( \overline{w}(P) \) and \( w(P) \) exist.

**Proof.** Since \( P \) is series-parallel, it follows that \( P \) is the sum of two posets. In particular, every maximal element in \( P \) is above any minimal one. If \( P \) has the greatest element then it is comparable with all other elements, so \( \overline{w}(P) \) and \( w(P) \) exist.

Suppose \( P \) has at least two maximal elements and let \( V \) denote the set of all maximal common lower bounds of the set of all maximal elements of \( P \). If \( V = \{p\} \) then \( p = \overline{w}(P) \). Indeed, if there is \( a \in P \) incomparable with \( p \) then there exists a maximal element of \( m \) such that \( a \not\leq m \). Take any maximal element \( m' \) such that \( a < m' \). Then \( \{a,p,m,m'\} \) is an \( N \)-poset, a contradiction. Suppose \( |V| \geq 2 \). If the elements of \( V \) have a unique greatest common lower bound, that is \( \inf V \), then one can show that it is \( \overline{w}(P) \). Assume \( \inf V \) does not exist. If \( V_1, V_2 \subseteq V, V_1 \cap V_2 \neq \emptyset \), and \( \inf V_1 \) and \( \inf V_2 \) would both exist then they would be comparable, because \( P \) is \( N \)-free. Moreover, we would have \( \inf(\inf V_1, \inf V_2) = \inf(V_1 \cup V_2) \). Since \( \inf V \) does not exist, one can find \( v_i, v_j \in V \) such that \( \inf(v_i, v_j) \) does not exist. Consider \( W = \{x \in P \mid x \geq v_i, v_j\} \). Note that this set contains all maximal elements of \( P \). The poset \( W \) cannot be connected, since otherwise any of its minimal elements would be below all maximal elements of \( P \) which is impossible by the choice of \( V \). Now pick two minimal elements \( w_k, w_l \) from different connected components of \( W \). It is easy to see that \( \{v_i, v_j, w_k, w_l\} \) form a 4-crown that contradicts the assumption that \( P \in X \).

Since there exists an element in \( P \) comparable with all other elements, \( w(P) \) exists as well. \( \square \)

** Lemma 2** Let \( P \in X \) and \( A, B \subseteq P \) be two antichains (one of which may be empty). Then every connected component of the subposet \( P_{A,B} \) belongs to \( X \).
Proof. Follows immediately from definitions, as the offending 4-crown in $P_{A,B}$ would be such in $P$. 

Proof of Theorem 1. It is easy to see that any retraction from $Q$ onto $P$ must map $Q_{A,B}$ to $P_{A,B}$. If $Q$ does not satisfy (R1) then $Q$ has an element that cannot be mapped anywhere by a retraction, due to the rule above. If $Q$ does not satisfy (R2) then, in $Q$, there is a path of comparable elements that are forced to be mapped to some disconnected $H$ and that connect two elements from different connected components of $H$. Hence, if $Q$ does not satisfy either (R1) or (R2) then there is no retraction from $Q$ onto $P$.

Suppose now that $Q$ satisfies both (R1) and (R2). We show that there is a retraction $r : Q \to P$. In order to define $r$ we need some constructions and notation.

In the following we write $\leq$ to denote $\leq Q$, this will cause no confusion. However sup and inf will always be calculated in $P$. Take $q \in Q$, and let $L_q$ be the set of maximal elements of $\{a \in P \mid a \leq q\}$, and $U_q$ the set of minimal elements of $\{b \in P \mid b \geq q\}$. Let $H_q = P_{L_q,U_q}$. Note that $H_q$ is non-empty.

It is not hard to show that if $H_q$ is not connected then there exists no set of the form $P_{A,B}$ such that $H_q$ is properly contained in $P_{A,B}$ and the elements from different connected components of $H_q$ are still disconnected in $P_{A,B}$. Since $Q$ satisfies (R2), there is at most one connected component $K_q$ of $H_q$ such that $q$ is connected to an element of $K_q$ in the way described in (R2). For every disconnected $H_q$, let us fix an arbitrary element $z(H_q)$ such that $z(H_q) = \overline{w}(K)$ for some connected component $K$ of $H_q$. Note that $z(H_q)$ actually depends on the set $H_q$, not on $q$. If $U_q$ and $L_q$ are both non-empty, denote by $X_q$ the set of of all $x \in P$ such that $x$ is minimal with the property that

1. $P_{L_q,U_q \cup \{x\}}$ is non-empty, and
2. $U_q \cup \{x\}$ is an antichain, and
3. sup$(x, u)$ does not exist for some $u \in U_q$.

Denote $P_{L_q,U_q \cup X_q}$ by $H_q'$. Throughout the proof we use the following property which can be easily derived from definitions: for every $x \in X_q$, either $P_{L_q,U_q \cup \{x\}} = P_{L_q,U_q}$ or, for all $u \in U_q$, sup$(x, u)$ does not exist. We need the following three claims in order to define the retraction.

Claim 1. If $H_q \neq \emptyset$ then $H_q' \neq \emptyset$.
We show that every minimal element of $H_q$ belongs to $H_q'$. Suppose, for contradiction, that there is a minimal element $a \in H_q$ and some $x \in X_q$ such that $a \not\leq x$. Then $\{a, b, x, u\}$ where $b$ is some minimal element of $H_q$ such that $b \leq x$, and $u$ is an arbitrary element in $U_q$, form an $N$-poset.

Claim 2. If $H_q$ is not connected then $H_q = H_q'$.
Both $U_q$ and $L_q$ contain more than one element. Take an arbitrary $x \in X_q$. If $x$ is above one maximal element of $P_{L_q,U_q}$ then it is above all elements of $P_{L_q,U_q}$ (or else one can easily find an $N$-poset), and then $P_{L_q,U_q} = P_{L_q,U_q \cup \{x\}}$. Otherwise, by the choice of $x$, there is $a$ in $P_{L_q,U_q \cup \{x\}}$. Take a maximal element $b$ in $P_{L_q,U_q}$ such that $a \not\leq b$, and any element $u \in U_q$. One can check that $\{a, b, x, u\}$ is an $N$-poset.

Claim 3. If $H_q$ is connected then so is $H_q'$.
It is shown above that all minimal elements of $P_{L_q,U_q}$ belong to $P_{L_q,U_q \cup X_q}$. If there is only
one minimal element then it ensures connectivity. Otherwise, for every pair \( a, b \) of distinct minimal elements, fix their minimal common upper bound \((P_{L_a U_a})v_{a,b}\). If every \( v_{a,b} \) is below all \( x \in X_q \) then we get connectivity. Assume, on the contrary, that there exist \( a, b \) and \( x \) such that \( v_{a,b} \not\in x \). Then \( \{a, b, v_{a,b}, x\} \) is a 4-crown, and there is no \( \text{inf}(a, b) \) (since \( a, b \) are minimal in \( P_{L_a U_a} \)). If there exists \( u = \text{sup} (x, v_{a,b}) \) then take any maximal element \( w \) in \( P_{L_a U_a} \) such that \( \text{sup}(w, x) \) does not exist and show that \( \{x, v_{a,b}, u, w\} \) is an \( N \)-poset. Indeed, \( w \) and \( u \) are incomparable. Obviously, \( x \not\in w \), and we have \( w \not\in u \), since otherwise \( u = \text{sup}(w, x) \). Thus, \( u = \text{sup}(x, v_{a,b}) \) does not exist, and we obtain a contradiction with \( P \in \mathcal{X} \). Claim 3 is proved.

Now we are ready to define the mapping \( r \). Let

\[
r(q) = \begin{cases} 
\overline{w}(P), & \text{if } L_q = U_q = \emptyset, \\
\overline{w}(H_q), & \text{if } L_q \neq \emptyset, U_q = \emptyset, \text{ and } H_q \text{ is connected}, \\
\overline{w}(H_q'), & \text{if } L_q = \emptyset, U_q \neq \emptyset, \text{ and } H_q \text{ is connected}, \\
\overline{w}(K_q), & \text{if } L_q \neq \emptyset, U_q \neq \emptyset, \text{ and } H_q \text{ is connected}, \\
\overline{w}(K_q'), & \text{if } L_q = \emptyset, H_q \text{ is disconnected, and } K_q \text{ is defined}, \\
z(H_q), & \text{if } H_q \text{ is disconnected and } K_q \text{ is not defined}.
\end{cases}
\]

Using Lemmas 1 and 2 it can be verified that \( r \) is well-defined, and that it is identity on \( P \). It remains to show that \( r \) is a homomorphism. Take \( q < q' \) in \( Q \) and prove that \( r(q) \leq r(q') \).

We distinguish seven cases depending on \( L_q, U_q, L_{q'}, U_{q'} \). Throughout the proof we use the following facts without explicit reference: 1) \( Q_{A,B} \) is non-empty whenever \( P_{A,B} \) is non-empty (since \( Q \) satisfies (R1)) and 2) if \( p_1, p_2 \in P \) and there is a path from \( p_1 \) to \( p_2 \) in \( Q_{A,B} \) then there is a path from \( p_1 \) to \( p_2 \) in \( P_{A,B} \) (this follows from the fact that \( Q \) satisfies (R2)). Moreover, in the second case \( p_1 \) and \( p_2 \) either are comparable or have a common upper or lower bound, since \( P \) is \( N \)-free.

**Case 1.** \( L_q = U_q = \emptyset \) or \( L_{q'} = U_{q'} = \emptyset \).

Suppose that \( L_q = U_q = \emptyset \), that is \( r(q) = \overline{w}(P) \). It follows that \( U_{q'} = \emptyset \). Assume that \( L_{q'} \neq \emptyset \), since otherwise trivially \( r(q) = r(q') \). If \( P_{L_{q'}, \emptyset} \) is not connected then it is easy to see that \( r(q) \) is below every element in \( P_{L_{q'}, \emptyset} \). If \( P_{L_{q'}, \emptyset} \) is connected then \( r(q) \leq r(q') \) because \( r(q) \) is comparable with every element in \( P \) and \( r(q) > r(q') \) would contradict the choice of \( r(q') \).

**Case 2.** \( L_q = U_{q'} = \emptyset \).

We have \( r(q) \leq \overline{w}(P) \leq \overline{w}(P) \leq r(q') \) (similarly to Case 1).

**Case 3.** \( L_q \neq \emptyset, U_q = \emptyset \).

It follows that \( L_{q'} \neq \emptyset \) and \( U_{q'} = \emptyset \).

**Subcase 3.1.** \( P_{L_{q'}, \emptyset} \) is not connected.

We have \( P_{L_{q'}, \emptyset} \not\subseteq P_{L_q, \emptyset} \).

Indeed, since \( q' \in Q_{L_q \cup L_{q'}, \emptyset} \), we know that \( P_{L_q \cup L_{q'}, \emptyset} \) is non-empty. Then there exists a maximal element \( u \in P_{L_{q'}, \emptyset} \) such that \( u \in P_{L_q, \emptyset} \). Then every maximal element \( v \) from any connected component \( K \) of \( P_{L_{q'}, \emptyset} \) such that \( u \not\in K \) must belong to \( P_{L_q, \emptyset} \), or otherwise one can easily find an \( N \)-poset there. Now if there is \( x \in P_{L_{q'}, \emptyset} \) such that \( a \not\leq x \) for some \( a \in L_q \) then take any maximal element \( v \) from a connected component of \( P_{L_{q'}, \emptyset} \) not containing \( x \), and take any element \( b \in L_q \). It is easy to see that \( \{a, b, x, v\} \) is an \( N \)-poset.

If every element \( x \in P_{L_{q'}, \emptyset} \) is connected with exactly the same elements in \( P_{L_q, \emptyset} \) as it is in \( P_{L_{q'}, \emptyset} \) then \( P_{L_q, \emptyset} = P_{L_{q'}, \emptyset} \). Indeed, otherwise one can choose \( a < u \) such that \( a \not\in P_{L_{q'}, \emptyset} \).
but \( u \in \mathbf{P}_{L_q, \emptyset} \). Take any element \( v \) from a connected component of \( \mathbf{P}_{L_q, \emptyset} \) not containing \( u \) and any element \( b \in L_q \) such that \( b \not\leq a \). Then \( \{a, b, u, v\} \) form an \( N \)-poset, a contradiction. So, in this case \( r(q) = r(q') \) because \( q \) and \( q' \) are connected in \( \mathbf{Q}_{L_q, \emptyset} = \mathbf{Q}_{L_q, \emptyset} \).

The argument from the previous paragraph also shows that if \( \mathbf{P}_{L_q, \emptyset} \) strictly contains \( \mathbf{P}_{L_q, \emptyset} \) then there exist \( a \) and \( u \) as described above and, moreover, every such \( a \) must lie below each element from \( \mathbf{P}_{L_q, \emptyset} \) (or else we can find an \( N \)-poset). So we now assume that \( \mathbf{P}_{L_q, \emptyset} \) is contained in one connected component of \( \mathbf{P}_{L_q, \emptyset} \). Now it is easy that if \( \mathbf{P}_{L_q, \emptyset} \) is connected then \( r(q) \leq r(q') \). If \( \mathbf{P}_{L_q, \emptyset} \) has at least two components then every element of \( L_q \) is above every element of \( L_q \), or else there is an \( N \)-poset (two incomparable elements from \( L_q \) and \( L_q \), and two maximal elements \( a, b \) such that \( u \in \mathbf{P}_{L_q, \emptyset} \) and \( v \) is from any other component of \( \mathbf{P}_{L_q, \emptyset} \). Therefore, \( L_q \), \( \mathbf{P}_{L_q, \emptyset} \), \( r(q) \) and \( r(q') \) are all in the same connected component \( K \) of \( \mathbf{P}_{L_q, \emptyset} \). Since \( r(q) \) is then below all elements of \( \mathbf{P}_{L_q, \emptyset} \), we have \( r(q) \leq r(q') \).

Subcase 3.2. \( \mathbf{P}_{L_q, \emptyset} \) is connected. Every maximal element of \( \mathbf{P}_{L_q, \emptyset} \) is a maximal element of \( \mathbf{P}_{L_q, \emptyset} \) or otherwise one can easily find an \( N \)-poset.

If \( \mathbf{P}_{L_q, \emptyset} \) is connected then \( r(q) \leq r(q') \) because of the above inclusion. If \( \mathbf{P}_{L_q, \emptyset} \) has at least two components then the argument is the same as in the previous subcase.

**Case 4.** \( U_q \neq \emptyset, L_q = \emptyset \).

Dual to Case 3.

From now on, we assume that \( U_q \neq \emptyset \) and \( L_q \neq \emptyset \), and at least one of \( L_q, U_q \) is non-empty.

Obviously, if \( u \leq l \) for some \( u \in U_q \) and \( l \in L_q \) then \( r(q) \leq r(q') \). Assume that \( l_1 \not\leq u_1, l_1 < u_2 \) for some \( l_1 \in L_q \) and \( u_1, u_2 \in U_q \). Then \( l_1 \not\leq r(q) \). If \( r(q) \) is incomparable with \( l_1 \) then \( \{r(q), l_1, u_1, u_2\} \) is an \( N \)-poset, a contradiction. So \( r(q) \leq l_1 \leq r(q') \). The argument is similar if there are \( l_1, l_2 \in L_q \) and \( u \in U_q \) such that \( l_1 < u \) and \( l_2 \not\leq u \).

In what follows we will always assume that either \( l < u \) for all \( l \in L_q \), \( u \in U_q \), or all such \( l \) and \( u \) are incomparable.

**Case 5.** \( L_q = \emptyset \) and \( U_q \neq \emptyset \).

Subcase 5.1 \( H_q \) is disconnected, and \( l < u \) for all \( l \in L_q \), \( u \in U_q \).

Suppose there are \( l_1', l_2' \in L_q \) belonging to two different connected components of \( H_q \). If \( r(q') \geq u \) for some element in \( u \in U_q \) then \( r(q) \leq r(q') \). If not then, since \( H_q \) is disconnected, there is \( u \in U_q \) such that \( r(q') \) and \( u \) are incomparable. Let \( l_1' \) not belong to the connected component of \( H_q \) containing \( r(q) \). Then \( r(q) \leq r(q') \), since otherwise \( \{r(q), l_1', u, r(q')\} \) is an \( N \)-poset.

So let \( L_q \) be contained in one connected component of \( H_q \). Then, for every \( a \in H_q \), either \( a \in H_q \), or \( a > b \) for every \( b \in H_q \), since otherwise \( \{a, b, x\} \) is an \( N \)-poset, where \( l \in L_q \) is such that \( l < a \), \( u \in U_q \) is such that \( u \) and \( a \) are incomparable, and \( x \) is a suitable maximal element of \( H_q \).

Assume first that \( H_q \subseteq H_q' \). Then, for any \( l \in L_q \), there is a path in \( Q_q \), from \( q \) to \( l \), such that every element on this path belongs to some \( \mathbf{Q}_{A, B} \subseteq H_q \). Therefore, both \( r(q) \) and \( r(q') \) belong to the same connected component \( K_q \) of \( H_q \), and \( r(q) = w(K_q) \). Then \( r(q) \) and \( r(q') \) are comparable. Moreover, by the definition of \( U_q \), \( r(q) \in H_q' \). If \( r(q) \in H_q' \) then \( r(q) \leq r(q') \), as \( r(q) \) and \( r(q') \) are both comparable with all elements of \( H_q' \). Assume that \( r(q) > r(q') \). Then \( r(q) \not\leq x \) for some \( x \in X_q \). Let \( A = \{a \mid a < r(q) \text{ and } r(q') \not\leq a\} \). This set is non-empty, since otherwise \( r(q') \) could not be above \( r(q') \). Let \( A' \) be the set of all maximal elements of \( A \). If some \( a \in A' \) satisfies \( a \not\leq x \) for some \( x \in X_q \) then \( \{a, r(q'), r(q), x\} \)
is an \( N \)-poset, a contradiction. If there exists \( \inf(r(q'), a) \) for all \( a \in A' \) then there exists \( c = \inf(A' \cup \{r(q')\}) \), and \( c \) would contradict the choice of \( r(q) \) because it is easy to show that \( c \) is comparable with all elements in \( H_q \). So \( \inf(r(q'), a) \) does not exist for some \( a \in A' \). Then \( \{a, r(q'), r(q), x\} \) is a crown contradicting \( P \in \mathcal{X} \) because the existence of \( \sup(r(q), x) \) would imply the existence of \( \sup(x, u) \) for every \( u \in U_q \) which, in turn, would contradict the definition of \( x \in X_q \). So if \( H_q' \subseteq H_q \) then \( r(q) \leq r(q') \).

Assume now that there is an \( H_q' \subseteq H_q \) such that \( a > b \) for every \( b \in H_q \). (Note that then \( H_q' \) is connected or every \( a \in H_q \) is above every \( b \in H_q \).) Consider minimal such elements. If there is only one such element \( a \) then \( r(q) \leq a \leq r(q') \). If there are more than two such elements \( a_1, \ldots, a_n \) then every pair \( (a_i, a_j) \) must have a supremum (or else \( \{a_i, a_j, y, z\} \), where \( y, z \) are from different connected components of \( H_q \), forms a crown that contradicts \( P \in \mathcal{X} \)). Then there exists \( c = \sup\{\{a_1, \ldots, a_n\}\} \). It is easy to check that \( c \) is comparable with all elements in \( H_q' \). Hence \( r(q) \leq c \leq r(q') \).

Subcase 5.2 \( H_q \) is disconnected, and \( l \) and \( u \) are incomparable for all \( l \in L_q', u \in U_q \).

If there are \( l \in L_q' \) and \( a \in H_q \) such that \( l > a \) then \( l > b \) for every \( b \) from any connected component of \( H_q \) not containing \( a \), or otherwise \( \{u, l, a, b\} \) is an \( N \)-poset for any \( u \in U_q \). So in this case \( r(q) \leq l \leq r(q') \). We may assume for the rest of this subcase that \( a \not\leq l \) for all \( l \in L_q' \) and \( a \in H_q \).

One can easily show that \( u \leq u' \) for every \( u \in U_q \) and \( u' \in U_q' \cup X_q' \) (or else one can find an \( N \)-poset).

For every \( l \in L_q' \) there exists \( b_1 \in H_q \) such that \( l < b_1 \) and \( u \neq b_1 \) for some \( u \in U_q \), since there is a path from \( l \) to \( u \) in \( Q_{\emptyset, U_q'} \). Thus \( b_1 > u \) for each \( u \in U_q \), or otherwise one can easily find an \( N \)-poset.

Take arbitrary \( u_1, u_2 \in U_q \). Since \( P \in \mathcal{X} \), there exists \( u = \sup(u_1, u_2) \). We know that \( u \leq u' \) for every \( u \in U_q \) and \( u' \in U_q' \cup X_q' \). If \( u > l \) for some \( l \in L_q' \) then one can take \( b_1 = u \), and, in fact \( u = \sup U_q \). We show that \( u \) is comparable with all elements of \( H_q' \).

Indeed, if \( u \) is incomparable with some \( a \in H_q' \) then, for each \( u_i \in U_q \) either \( \{u, a, u_i, l\} \) is an \( N \)-poset or \( a > u_i \). In the latter case, \( a \) must be comparable with \( u = \sup U_q \). Now we have \( r(q) < u \leq r(q') \) regardless whether \( u \in H_q' \) or not.

We may assume now that \( l \not\leq u \) for each \( l \in L_q' \).

Assume that \( H_q' \) is disconnected. If there is \( x \in H_q' \) such that \( u \not\leq x \) then there exists \( c \in H_q' \) such that \( c > x \) and \( c > u \), and then one can always find a maximal element \( y \in H_q' \) such that \( \{x, u, c, y\} \) is an \( N \)-poset. If \( u \leq x \) for all \( x \in H_q' \), then \( r(q) \leq r(q') \).

Assume that \( H_q' \) is connected. We know that \( r(q') \not\leq u \). Assume that \( r(q') \) and \( u \) are incomparable and derive a contradiction. Since \( r(q') \) and \( u \) are connected by a path in \( Q_{\emptyset, U_q'} \), they have a common bound in \( P_{\emptyset, U_q'} \). It is easy to show that if they have a common lower bound then one can prove as in the first paragraph of this subcase that \( r(q) \leq r(q') \). So assume that \( r(q') \) and \( u \) have no common lower bound, and there is \( b \in H_q' \) such that \( u < b \) and \( r(q') < b \). Let \( B \) be the set of all minimal \( b \) with this property. It is not hard to verify that there cannot be only one such minimal element. Any \( b \in B \) must belong to \( H_q' \). Indeed, if \( b \not\leq x \) for some \( x \in X_q' \) then \( \{b, x, u, r(q')\} \) form a crown contradicting the assumption that \( P \in \mathcal{X} \). If \( \sup(b_i, b_j) \) exists for any pair \( b_i, b_j \in B \) then there exists \( \sup B \). It is easy to check this element belongs to \( H_q' \), and it is comparable with every element of this set. This is a contradiction with the choice of \( r(q') \). So \( \sup(b_i, b_j) \) does not exist for some \( b_i, b_j \in B \). Then \( \{u, r(q'), b_i, b_j\} \) is a crown contradicting the assumption that \( P \in \mathcal{X} \). Thus we have \( r(q) \leq u \leq r(q') \).
Subcase 5.3 $H_q$ is connected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$.

In this case $r(q) = w(H_q)$. We assume that $r(q) \in H_{q'}$, since otherwise $r(q) \leq l \leq r(q')$ for some $l \in L_{q'}$.

Assume that $H'_{q'}$ is connected, and $r(q)$ is incomparable with some $x \in X_{q'}$, that is, $r(q) \notin H'_{q'}$. Note that $\sup(x, r(q))$ does not exist. If $r(q)$ and $r(q')$ are incomparable then, for any $u' \in U_{q'}$, $\{r(q), r(q'), u', x\}$ is an $N$-poset. Assume, for contradiction that $r(q) > r(q')$. Let $A$ be the set of all $a$ minimal with the following property: $a < r(q)$ and $a \neq r(q')$. If $\inf(a, r(q'))$ exists for each $a \in A$ then there exists $\sup(A \cup \{r(q')\})$, and this element is comparable with all elements in $H_q$, a contradiction with the choice of $r(q)$. So $\inf(a(r(q'))) does not exist for some $a \in A$. Then $\{a, r(q'), r(q), x\}$ is a crown contradicting $P \in X$.

Consequently, if $H'_{q'}$ is connected then $r(q) \in H'_{q'}$. Then, as $r(q') = w(H'_{q'})$, $r(q)$ and $r(q')$ are incomparable. Assume, for contradiction, that $r(q) > r(q')$. Let $B$ be the set of all elements $b$ minimal with the following property: $b < r(q)$ and $b \neq r(q')$. Similarly, let $C$ be the set of all elements $c \in H'_{q'}$ minimal with the following property: $c > r(q')$ and $c \neq r(q)$. By definitions of $r(q)$ and $r(q')$, both $B$ and $C$ are non-empty. Arguing as above, one can show that there are $b \in B$ and $c \in C$ such that neither $\inf(b, r(q'))$ nor $\sup(c, r(q))$ exist. Then $\{b, r(q'), c, r(q)\}$ is a crown contradicting $P \in X$.

Assume now that $H_q$ is disconnected. Note that $L_{q'}$ has at least two elements. Let $b$ be a minimal element from the connected component of $H_{q'}$ containing $r(q)$ and $a$ a minimal element from any other connected component. In $H_q$, consider a maximal (with respect to inclusion) antichain $D$ such that $L_{q'} \subseteq D$. It is easy to show that, for every $d \in D$, both $d < a$ and $d < b$ holds, or else $P \notin X$. The standard argument shows that there must be $d_1, d_2 \in D$ such that $\inf(d_1, d_2)$ does not exist. By the choice of $a$ and $b$, $\sup(a, b)$ does not exist either. Then $\{a, b, d_1, d_2\}$ is a crown contradicting $P \in X$.

Subcase 5.4 $H_q$ is connected, and $l$ and $u$ are incomparable for all $l \in L_{q'}$, $u \in U_q$.

Assume that $H_q$ is disconnected. If $r(q) \leq l$ for some $l \in L_{q'}$ then $r(q) \leq r(q')$. Assume now that $r(q)$ is incomparable with each $l \in L_{q'}$. Since, in $Q_{\emptyset, U_{q'}}$, there is a path from $r(q)$ to any such $l$, we see that $r(q)$ and $l$ have a common bound in $P_{\emptyset, U_{q'}}$. If, for some $l \in L_{q'}$, there is $b$ such that $b > l$ and $b > r(q)$ then using disconnectedness of $H_q$, it is easy to prove that $r(q) < a$ for every $a \in H_{q'}$, in particular, $r(q) < r(q')$. Assume now that, for each $l \in L_{q'}$, $r(q)$ and $l$ have only lower bounds in common. It follows, in particular, that $\sup(l, r(q))$ exists for no $l \in L_{q'}$. Fix an arbitrary $l \in L_{q'}$ and consider the set $A$ consisting of all maximal common lower bounds of $r(q)$ and $l$. Extend $A$ to a maximal antichain $D$ in $H_q$. Arguing as in the previous subcase, one can derive a contradiction with $P \in X$.

Assume now that $H'_{q'}$ is connected. We know that $r(q') \not\leq r(q)$. Assume that these elements are incomparable and derive a contradiction. If $r(q) \not\leq x$ for some $x \in X_{q'}$ then $\{r(q), r(q'), x, u'\}$ is an $N$-poset for every $u' \in U_{q'}$. So we have $r(q) \in P_{\emptyset, U_{q'} \cup X_{q'}}$. Arguing as above, one can prove that if $\inf(r(q), r(q'))$ exists then it must be $r(q)$. Similarly, if $\sup(r(q), r(q'))$ exists then it must be $r(q')$. So by our assumption neither $\inf(r(q), r(q'))$ nor $\sup(r(q), r(q'))$ exists. We know that $r(q)$ and $r(q')$ have a common upper or lower bound in $P_{\emptyset, U_{q'} \cup X_{q'}}$, since they are connected by a path in $Q_{\emptyset, U_{q'}}$. Assume that none of the common bounds belongs to $P_{\emptyset, U_{q'} \cup X_{q'}}$. Then all their common bounds are upper. Take a minimal upper bound $c \in P_{\emptyset, U_{q'}}$. There is $x \in X_{q'}$ such that $c$ and $x$ are incomparable. Then, by construction of $X_{q'}$, $\sup(c, x)$ does not exist, and $\{r(q), r(q'), x, c\}$ form a crown contradicting $P \in X$. So $r(q)$ and $r(q')$ do have common bounds in $P_{\emptyset, U_{q'} \cup X_{q'}}$. Now depending on whether they have lower or upper bounds, one can find, using the “antichain” method above, a crown
with the top or the bottom elements $r(q)$ and $r(q')$ such that this crown contradicts the assumption $P \in \mathcal{X}$.

**Case 6.** $L_q \neq \emptyset$ and $U_{q'} = \emptyset$.

Subcase 6.1 $H_{q'}$ is disconnected, and $l < u$ for all $l \in L_{q'}$, $u \in U_{q'}$. The argument dual to the one from the first two paragraphs of Subcase 5.1 shows that it is enough to consider the case when $U_q$ is contained in one connected component of $H_{q'}$, and, for every $a \in H_q$, either $a \in H_{q'}$ or $a < b$ for every $b \in H_{q'}$.

Assume first that $H'_q \subseteq H_{q'}$. Then, for any $u \in U_q$, $\psi_{H_{q'}}(u, q')$ is true, and therefore $r(q)$ and $r(q')$ are in the same connected component $K'_q$ of $H_{q'}$, and $r(q') = w(K'_q)$. Then $r(q)$ and $r(q')$ are comparable. If $r(q') \leq u$ for each $u \in U_q$ then, since $r(q') = w(K'_q)$, we have $r(q') > u$ for some $u \in U_q$, and $r(q) < r(q')$. So let $r(q') \in H'_q$. Then $H_q$ is connected and $r(q) = w(H'_q)$. Assume, for contradiction, that $r(q) > r(q')$. Let $A$ be a maximal (with respect to inclusion) antichain in $K'_q$ such that $U_q \subseteq A$, and let $B = \{b \in A \mid r(q) \leq b\}$. If $A = B$ then it is easy to show that $r(q)$ is comparable with each element in $K'_q$, a contradiction with the choice of $r(q')$. Assume that $A \neq B$. If, for every $x \in A \setminus B$ and every $u \in U_q$, there exists $\sup_x(u, u)$ then there exists $c = \sup(U_q \cup (A \setminus B))$. If $b \leq c$ for every $b \in B$ then it is easy to show that $c$ is comparable with each element in $K'_q$, a contradiction with the choice of $r(q')$. If $b \not\leq c$ for some $b \in B$ then, for any $x \in A \setminus B$, $\{x, r(q), b, c\}$ is an $N$-poset. Hence there are $x \in A \setminus B$ and $u \in U_q$ such that $\sup_x(u, u)$ does not exist. Then $x \in X_q$. Since $r(q) \not\leq x$, we obtain a contradiction with the choice of $r(q)$. Thus if $H'_q \subseteq H_{q'}$ then $r(q) \leq r(q')$.

Assume now that there is $a \in H'_q$ such that $b < a$ for every $b \in H_{q'}$. If all elements in $H'_q$ are such then, obviously, $r(q) < r(q')$. If not then $H'_q$ is connected and then $r(q) = w(H'_q)$.

Note that there exist $l \in L_q$ and $l' \in L_{q'}$ such that $l < l'$. Take any element $x$ from a connected component of $H_{q'}$ not containing $U_q$. The element $l$ ensures that $x \in X_q$. Thus we have $r(q) \leq x$, and $r(q)$ must be below all elements in $H_{q'}$, in particular, $r(q) < r(q')$.

Subcase 6.2 $H_{q'}$ is disconnected, and $l$ and $u$ are incomparable for all $l \in L_{q'}$, $u \in U_{q'}$. Take arbitrary $l$ and $u$. Since $l$ and $u$ are connected by a path in $Q_{L_q, 0}$, they have a common bound in $P_{L_q, 0}$. Assume that some $l$ and $u$ have a common upper bound $b \in P_{\Theta_{L_q, 0}}$. Let $l' \not\leq b$ for some $l' \in L_{q'}$. Then either $b < a$ for every $a \in H_q$ (and then $r(q) \leq r(q')$) or there is a from a suitable connected component of $H_{q'}$ such that $\{l, l', a, b\}$ is an $N$-poset, a contradiction. Assume now that $l' < b$ for all $l' \in L_{q'}$. Again, either $u < a$ for every $a \in H_q$ (and then $r(q) \leq r(q')$) or one can find an $N$-poset in $P$, a contradiction.

Assume for the rest of this subcase that no $l \in L_{q'}$ and $u \in U_q$ have a common upper bound. Fix $l \in L_{q'}$. For any $u \in U_q$, there is a common lower bound $d_u$ in $P_{L_q, 0}$ of $l$ and $u$. Choose $d_u$ to be minimal in $P_{L_q, 0}$. If $d_u \not\leq u_1$ for some $u_1 \in U_q$ then $\{d_u, d_u, u_1, l\}$ is an $N$-poset. Hence $d_u \in H_q$, and, moreover, $P_{L_q, H_q \cup \{l\}}$ is non-empty. If $H_q$ is connected then, by the definition of $X_q$, there is $x \in X_q$ such that $x \leq l$, and then $r(q) \leq x \leq l \leq r(q')$. If $H_q$ is disconnected then $a < l$ for every $a \in H_q$, and then $r(q) < l < r(q')$, or else one can choose $a \in H_q$ so that $\{a, d_u, u, l\}$ is an $N$-poset.

Subcase 6.3 $H_{q'}$ is connected, and $l < u$ for all $l \in L_{q'}$, $u \in U_q$. In this case $r(q') = w(H_{q'})$. We assume that $r(q') \in H_q$, since otherwise $r(q) \leq u \leq r(q')$ for some $u \in U_q$.

Let $H_q$ be connected. If $r(q)$ and $r(q')$ are incomparable then there is $x \in X_q$ such that $r(q') \not\leq x$, and $\{r(q), r(q'), x, u\}$ is an $N$-poset for any $u \in U_q$. So $r(q)$ and $r(q')$ are comparable. Assume, for contradiction, that $r(q) > r(q')$. The argument similar to the one from the second paragraph of the previous subcase shows that this leads to a contradiction.
with the choice of \( r(q') \)

Let \( H_q \) be disconnected. It is easy to see that \( r(q) \) and \( r(q') \) belong to the same connected component of \( H_q \). Then the proof is similar to (the dual of) the last paragraph of subcase 5.3.

**Subcase 6.4** \( H_{q'} \) is connected, and \( l \) and \( u \) are incomparable for all \( l \in L_{q'}, u \in U_{q'} \).

We have \( r(q') = \mathfrak{w}(H_{q'}) \), and we may assume that \( r(q') \) is in comparable with \( u \) for each \( u \in U_{q'} \), since otherwise we immediately have \( r(q) < r(q') \).

Take \( l \in L_{q'} \) and \( u \in U_{q'} \). Since \( u \) and \( l \) are connected by a path in \( \mathcal{Q}_{L_{q'}, \emptyset} \), they have a common bound in \( \mathcal{P}_{L_{q'}, \emptyset} \).

Assume they have a common upper bound \( b \). Then either \( a < l \) for every \( a \in H_{q'} \), and then \( r(q) < r(q') \), or \( u < b \) for every \( u \in U_{q'} \). Similarly, either \( u < c \) for every \( c \in H_{q'} \), and then \( r(q) < r(q') \), or \( l < b \) for every \( l \in L_{q'} \). Assume that \( b \) is above each element in \( U_{q'} \cup L_{q'} \). Then \( b \) is comparable with \( r(q') \). If \( l \leq r(q') \) then, of course, \( r(q) < r(q') \). So let \( r(q') < b \) for every such \( b \). Let \( B \) be the set of all minimal elements of \( \{ b \mid r(q') < b \text{ and } u < b \text{ for all } u \in U_{q'} \} \).

It is easy to check that every \( a \) such that \( r(q') \leq a \) is comparable with some element of \( B \). If there exist \( \sup(b_1, b_2) \) for all \( b_1, b_2 \) in \( B \) then there exists \( \sup B \), and this element contradicts the choice of \( r(q') \). So \( \sup(b_1, b_2) \) does not exist for some \( b_1, b_2 \) in \( B \). Since \( \mathcal{P} \in \mathcal{X} \), there is \( \inf(r(q'), u) \) for every \( u \in U_{q'} \). As in subcase 6.2, we can choose \( d_u \leq \inf(r(q'), u) \) so that \( d_u \in H_q \) and then derive that \( r(q) < r(q') \).

**Case 7.** \( L_q \neq \emptyset \) and \( U_{q'} \neq \emptyset \).

**Subcase 7.1** The inequality \( l < u \) holds for all \( l \in L_{q'} \), \( u \in U_{q'} \).

Assume, for contradiction, that \( r(q) \) and \( r(q') \) are incomparable. Then, since \( \mathcal{P} \) is \( N \)-free, for every \( u \in U_{q'} \) and \( l' \in L_{q'} \), we have \( r(q) > l' \) or \( r(q') < u \). For the same reason, if \( r(q) > l' \) for some \( l' \in L_{q'} \) then \( r(q) > l' \) for each \( l' \in L_{q'} \), and if \( r(q') < u \) for some \( u \in U_{q'} \) then \( r(q') < u \) for each \( u \in U_{q'} \). Hence, \( r(q) \in H_{q'} \) or \( r(q') \in H_{q'} \), or both. It is easy to verify that \( r(q) \in H_{q'} \) implies \( r(q) \in H_{q'}^{l'} \) and \( r(q') \in H_{q} \) implies \( r(q') \in H_{q}^{l'} \).

If \( u \leq r(q') \) then, obviously, \( r(q) \leq r(q') \). Assume that \( r(q') \) is incomparable with some \( u \in U_{q'} \). Then \( r(q') \) is incomparable with each \( u \in U_{q'} \) and we have \( r(q) \in H_{q'}^l \). Since \( \mathcal{P} \) is \( N \)-free, it follows that \( u < u' \) for each \( u \in U_{q'} \), \( u' \in U_{q'} \), that is \( U_{q'} \subseteq H_{q'} \), and, moreover, \( U_{q'} \subseteq H_{q}^{l'} \). Since, by definition, \( r(q') \) is comparable with all elements from its connected component of \( H_{q'}^{l'} \), we conclude that \( H_{q'} \) is disconnected, and the elements \( r(q) \) and \( r(q') \) belong to different connected components of \( H_{q'}^{l'} \). Moreover, \( \sup(u, r(q')) \) does not exist for any \( u \in U_{q'} \). There exists \( a \in H_q \) such that \( a \not\in H_{q'} \), since otherwise, by definition, \( r(q) \) and \( r(q') \) would belong to the same connected component of \( H_{q}^{l'} \). Then \( l' \not\leq a \) for some \( l' \in L_{q'} \). If \( a < l' \) then \( a \in \mathcal{P}_{L_{q'}, U_{q'} \cup \{ r(q') \}} \) and \( r(q) \leq x \leq r(q') \) for some \( x \in X_{q'} \), a contradiction with our assumption that \( r(q) \) and \( r(q') \) are incomparable. If \( a \not\leq l' \) incomparable then it is easy to show that \( l' \in \mathcal{P}_{L_{q'}, U_{q'} \cup \{ r(q') \}} \), so \( r(q) \leq r(q') \), a contradiction again. We conclude that \( r(q') \) cannot be incomparable with each \( u \in U_{q'} \), and so \( r(q') \in H_{q'}^{l'} \). Similarly, one can show that \( r(q) \in H_{q'}^{l'} \).

Then both \( H_q \) and \( H_{q'} \) are disconnected, and \( r(q) \) and \( r(q') \) belong to different connected components of both \( H_q \) and \( H_{q'} \), since otherwise \( r(q) \) and \( r(q') \) would be comparable. It follows that \( \mathcal{P}_{\Theta_{L_{q'}, U_{q'}}} \) is non-empty and disconnected, and, of course, \( \mathcal{P}_{L_{q'}, U_{q}} \subseteq H_q \cap H_{q'} \). As we have already noticed when defining \( H_q \), such an inclusion implies \( \mathcal{P}_{\Theta_{L_{q'}, U_{q}}} = H_q = H_{q'} \).

But then, by definition, \( r(q) \) and \( r(q') \) would belong to the same connected component of this set, and so would be comparable. This is a contradiction with our assumption that \( r(q) \) and \( r(q') \) are incomparable.
Assume now, for contradiction, that \( r(q) > r(q') \). In this case, we again have \( r(q') \in H_q \)
and \( r(q) \in H_{q'} \). If it is easy to show that if \( x' \in X_{q'} \) then either \( x' \in X_q \) or \( u < x' \) for some \( u \in U_q \). In both cases \( r(q) \) must be below \( x' \), and so \( r(q) \in H'_{q'} \). We have \( r(q') \in H'_{q'} \), as \( r(q') < r(q) \). Now choose \( A \) and \( B \) as in the second paragraph of subcase 6.1, and repeat the argument from that paragraph to derive a contradiction.

**Subcase 7.2** \( H_{q'} \) is disconnected, and \( l \) and \( u \) are incomparable for all \( l \in L_{q'} \), \( u \in U_q \).

Note that \( l < l' \) for every \( l \in L_q \), \( l' \in L_{q'} \), and \( u < u' \) for every \( u \in U_q \), \( u' \in U_{q'} \) because the poset \( P \) is \( N \)-free.

Take arbitrary \( l \) and \( u \). Since \( l \) and \( u \) are connected by a path in \( Q_{L_q,H_{q'}} \), they have a common bound in \( P_{\Theta_{L_q,H_{q'}}} \). The rest of the proof for this subcase is identical to subcase 6.2.

**Subcase 7.3** \( H_{q'} \) is disconnected, and \( l \) and \( u \) are incomparable for all \( l \in L_{q'} \), \( u \in U_q \).

Similar to subcase 6.4.

The theorem is proved. \( \Box \)

## 4 Proof of Theorem 2

(4) \( \Rightarrow \) (3) : trivial.

(3) \( \Rightarrow \) (5) : this follows immediately from Theorem 3.2 of [10].

(5) \( \Rightarrow \) (6) : trivial.

(6) \( \Rightarrow \) (2) : suppose that (2) does not hold. This means that \( P \) retracts onto a poset \( R \) which has a connected idempotent subalgebra which is not simply connected: in fact, this subalgebra is a 4-crown in every case. Indeed, for \( R = 2 + 2 \), it is \( R \) itself; the 4 top elements of \( 2 + 2 + 2 \) do the trick, and similarly for \( 1 + 2 + 2 + 2 \). For \( 2 + 2 + 2 + 1 \), the bottom 4 elements will do, and finally the 4 middle elements do the job for \( 1 + 2 + 2 + 2 + 2 + 1 \). In each case, it is easy to check, by using Definition 4, that the selected 4-crown is indeed an idempotent subalgebra. Thus \( P \) retracts onto a poset \( R \) which has an idempotent subalgebra isomorphic to a 4-crown. By results in [8] this implies that the 4-crown is a retract of an idempotent subalgebra of \( P \), call it \( Q \). It is easy to check that the connected components of an idempotent subalgebra are themselves idempotent subalgebras, so we may assume that \( Q \) is connected. But then the fundamental group of \( Q \) retracts onto the fundamental group of the 4-crown, which is \( Z \), a contradiction.

(2) \( \Rightarrow \) (1) : we prove the contrapositive by induction on the size of \( P \). If \( P = P_1 \cup P_2 \) then \( P_1 \) or \( P_2 \) contains the offending crown, so we are done by induction. Otherwise we have that \( P = P_1 + P_2 \). Suppose first that \( P_1 \) contains an offending crown. Then, by induction hypothesis, \( P_1 \) retracts onto one of the 5 posets in the list. Since any combination of retractions of \( P_1 \) and \( P_2 \) is a retraction of \( P \), it follows that it is enough to prove the result assuming that \( P_1 \) actually is one of the 5 posets in the list. If \( P_1 \) is neither \( 2 + 2 \) nor \( 1 + 2 + 2 + 2 \) then \( P \) retracts onto \( 2 + 2 + 2 + 1 \) or \( 1 + 2 + 2 + 2 + 2 + 1 \) by retracting \( P_2 \) to a point, so we are done. Now suppose that \( P_1 = 1 + 2 + 2 + 2 \). Then the offending crown must consist of the top four elements, which means that \( P_2 \) must have at least 2 minimal elements; if these have an upper bound then \( P_2 \)
retracts onto $2 + 1$, so $P$ retracts onto $1 + 2 + 2 + 2 + 1$; otherwise (i.e., if no pair of minimal elements have an upper bound), $P_2$ retracts onto $2$ and so $P$ retracts onto $1 + 2 + 2 + 2$, which retracts onto $1 + 2 + 2 + 2$. If $P_2$ contains an offending crown the proof is dual. So we may now suppose that $P_1$ has 2 maximal elements with no meet and $P_2$ has 2 minimal elements with no join.

**Claim.** Let $Q$ be a series-parallel poset of height at least 2, and let $x, y$ be maximal elements in $Q$ that have a lower bound but no meet. Then $Q$ retracts onto $1 + 2 + 2$.

**Proof of Claim.** We use induction on the size of $Q$. Clearly the result holds for small posets, and if $Q = Q_1 \cup Q_2$ the result follows immediately by induction. So suppose that $Q = Q_1 + Q_2$. Then $\{x, y\} \subseteq Q_2$. Since $x$ and $y$ have no infimum in $Q$ they have no infimum in $Q_2$. If $\{x, y\}$ has a lower bound in $Q_2$ then by induction it retracts onto $1 + 2 + 2$ and so does $Q$, else $Q_2$ retracts onto 2 because $Q_2$ is $N$-free. Since $x$ and $y$ have no meet in $Q$ it follows that $Q_1$ has at least 2 maximal elements, and these have a lower bound. Hence $Q_1$ retracts onto $1 + 2$ and so $Q$ retracts onto $1 + 2 + 2$.

By the claim and its dual, $P_1$ retracts onto $2$ or $1 + 2 + 2$, and $P_2$ retracts onto $2$ or $2 + 2 + 1$. Combining the 4 cases yields the desired result.

(1) $\Rightarrow$ (4) : we must show that every poset $P \in \mathcal{X}$ has TSI operations of every arity $k \geq 2$. Let $K = K(P)$ denote the following poset, as defined in [10] (see also [9, 19]): a subset $C$ of a poset $P$ is (weak) convex if for all $c, c' \in C$, if $c \leq x \leq c'$ then $x \in C$. $K$ is the poset of all non-empty convex sets in $P$ ordered as follows: $X \leq Y$ if and only if for every $x \in X$ there exists $y \in Y$ such that $x \leq y$, and for every $v \in Y$ there exists $u \in X$ such that $u \leq v$. The embedding of $P$ into $K$ sends $p$ to $\{p\}$. By Proposition 1 in [19] and the last part of the proof of Theorem 3 in [19] (see also remark (1) on page 95 of [4]), $P$ admits TSI operations of all arities if and only if it admits one of arity $|P|$, if and only if $P$ is a retract of $K$. Hence, it suffices to prove that $P$ is a retract of the poset $K$. Here we invoke Theorem 1.

(R1) Let $\langle A, B \rangle$ be a pair of antichains in $P$ such that $K_{A,B} \neq \emptyset$; we must show that $\mathcal{P}_{A,B} \neq \emptyset$. But it is immediate by the definition of $K$ and the embedding of $P$ in $K$ that if $X \in K_{A,B}$ then $X \subseteq \mathcal{P}_{A,B}$.

(R2) Let $H = \mathcal{P}_{A', B'}$ be non-empty, let $p_1, p_2 \in H$ be such that there exists a path from $\{p_1\}$ to $\{p_2\}$ in $K$ such that every element in this path lies in some $K_{A,B}$ with $\mathcal{P}_{A,B} \subseteq H$. We must show that $p_1$ and $p_2$ are connected via a path in $H$ itself. More precisely, consider a path

$$\{p_1\} = X_0, X_1, \ldots, X_n = \{p_2\}$$

in $K$ such that for each $i = 1, \ldots, n - 1$, there are some antichains $A_i, B_i$ in $P$ such that $X_i \in K_{A_i, B_i}$ and $\mathcal{P}_{A_i, B_i} \subseteq H$. As in the proof of property (R1) above, we have $X_i \subseteq \mathcal{P}_{A_i, B_i} \subseteq H$ for all $i$. By definition of comparability in $K$, we may find a sequence of elements $x_1 \in X_1, x_2 \in X_2, \ldots$ such that $p_1, x_1, x_2, \ldots, x_{n-1}, p_2$ is a path in $H$ from $p_1$ to $p_2$.

$\square$
5 Proof of Theorem 3

For any poset $P'$, let $\text{Ext}(P')$ denote the following problem: given a poset $Q$ and a partial mapping $f$ from $Q$ to $P'$, does $f$ extend to a homomorphism from $Q$ to $P'$. We will use the following result in our $\text{NP}$-completeness proof. This result is more or less folklore, but, for completeness, we present it with a proof.

**Proposition 1** For any finite poset $P'$, the problems $\text{PoRet}(P')$ and $\text{Ext}(P')$ are polynomial time equivalent.

**Proof.** Obviously, $\text{PoRet}(P')$ is a subproblem of $\text{Ext}(P')$, with $f$ being the identity mapping defined on $P'$. We now show that $\text{Ext}(P')$ reduces to $\text{PoRet}(P')$ in polynomial time. Let $(Q, f)$ be an instance of $\text{Ext}(P')$. We may without loss of generality assume that $Q \cap P' = \emptyset$. Let $Q'$ be a poset that is a disjoint union of $Q$ and $P'$, and $f'$ a partial mapping from $Q'$ to $P'$ that coincides with $f$ on $Q$ and is identity on $P'$. Obviously, the instances $(Q, f)$ and $(Q', f')$ are equivalent. Let $\varrho'$ be the transitive closure of the following relation $\varrho$ on $Q'$: $(x, y) \in \varrho$ if and only if $f'(x) = f'(y)$ or $x \leq y$ in $Q'$. It is easy to see that $\varrho'$ can be computed from $(Q, f)$ in polynomial time. Moreover, if $(p_1, p_2) \in \varrho'$ for some $p_1, p_2 \in P'$ such that $p_1 \not\leq p_2$ in $P'$ then $f$ does not extend to a homomorphism. Assume that this condition is not satisfied. Then all elements of $P'$ belong to different classes of the equivalence relation $\epsilon = \varrho' \cap (\varrho')^{-1}$. Now let $Q''$ be a complete set of representatives of $\epsilon$-classes such that $P' \subseteq Q''$, and let $\sqsubseteq$ be the partial order on on $Q''$ induced by $\varrho'$. Then the poset $Q'' = (Q'', \sqsubseteq)$ is an instance of $\text{PoRet}(P')$. It is straightforward to check that $Q''$ retracts onto $P'$ if and only if $f$ extends to a homomorphism. Since $Q''$ can computed from $(Q, f)$ in polynomial time, the result follows. □

**Proof of Theorem 3.** It is easy to see that if $P$ is not connected then $Q$ retracts onto $P$ if and only if different connected components of $P$ are not connected in $Q$ and, for every connected component of $P$, the connected component of $Q$ containing it retracts onto it. Since these conditions can be checked in polynomial time, we may from now on assume that $P$ is connected.

We begin with the $\text{NP}$-completeness part. Suppose that $P \not\in \mathcal{X}$. Theorem 2 implies that $P$ has no Taylor polymorphism. Corollary 10 and Theorem 4 of [9] imply that that if $P$ has no Taylor polymorphism then $\text{Ext}(P)$ is $\text{NP}$-complete. By Proposition 1, $\text{PoRet}(P)$ is $\text{NP}$-complete.

Assume now that $P$ satisfies the conditions of Theorem 2. In particular, $P$ has TSI polymorphisms of every arity $k \geq 2$. Then Proposition 1 above and Proposition 9 [9], imply that $\text{PoRet}(P)$ is polynomial-time equivalent to a certain constraint satisfaction problem (see [9]) whose tractability follows from [2] (or [4]). □

6 Conclusion

We have studied the poset retraction problem for series-parallel posets. We have obtained a full complexity classification for such problems and have characterized in several ways those posets for which the problem is tractable. It is an interesting research problem to find out
whether some of the conditions of Theorem 2 (in particular, conditions (3) and (6)) determine tractable poset retraction problems for general posets.

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References


