Master Degree in Economics and Finance

Partial Adjustment in Policy
Functions of Structural Models of Capital Structure
Author: Mattia Bongini

Director: Filippo Ippolito

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ABSTRACT IN ENGLISH:
We present a tradeoff model of capital structure to investigate the sources of adjustment costs and study how firms' financing decisions determine partial adjustment toward target leverage ratios. The presence of market imperfections, like taxes and collateral constraints, is shown to play a decisive role in the behavior of the policy function of capital and leverage. By means of a contraction argument, we are able to show the existence of a target leverage towards which optimal leverage converges with a speed of adjustment that depends on a firm marginal productivity of capital. Our predictions are consistent with the empirical literature regarding both the magnitude of the speed of adjustment and the relationship between leverage ratios and the business cycle.

ABSTRACT IN CATALAN:
Presentem un model de compensació d'estructura de capital per investigar les fonts dels costos d'ajustament i estudiem com les decisions de finançament de les empreses determinen un ajust parcial a les ràtios d'apalancament objectiu. La presència d'imperfeccions del mercat, com ara impostos i restriccions de garantia, demostra un paper decisiu en el comportament de la funció política del capital i l'apalancament. Mitjançant un argument de contracció, podem demostrar l'existència d'un apalancament objectiu cap al qual convergeix un palanquejament òptim amb una velocitat d'ajustament que depèn d'una productivitat marginal ferma del capital. Les nostres predicccions són coherents amb la literatura empírica quant a la magnitud de la velocitat d'ajust i la relació entre els coeficients d'apalancament i el cicle econòmic.
Partial Adjustment in Policy Functions of Structural Models of Capital Structure

Mattia Bongini

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Abstract

We present a tradeoff model of capital structure to investigate the sources of adjustment costs and study how firms’ financing decisions determine partial adjustment toward target leverage ratios. The presence of market imperfections, like taxes and collateral constraints, is shown to play a decisive role in the behavior of the policy function of capital and leverage. By means of a contraction argument, we are able to show the existence of a target leverage towards which optimal leverage converges with a speed of adjustment that depends on a firm marginal productivity of capital. Our predictions are consistent with the empirical literature regarding both the magnitude of the speed of adjustment and the relationship between leverage ratios and the business cycle.

1 Introduction and review of the literature

The study of how do firms choose and adjust their capital structure has been the focus of heated debate in financial economics since the celebrated Modigliani-Miller irrelevance principle [22]. The literature contains little consensus on how financial constraints and market imperfections impact the value of firms or their investment and financing decisions. On the theoretical side, supporters of the pecking-order theory of capital structure (introduced in [7] and modified in its present form in [24]), for instance, claim that firm have no preference about adjusting their leverage ratios to a particular level, since that quantity only reflects firms’ past profitability and future investment opportunities. Instead, the standard trade-off theory, first proposed in [19], predicts that the tax benefits of debt and the costs of financial distress entail the existence of an optimum capital structure, and firms select their target debt/equity ratios by balancing the marginal costs and benefits of leverage.

For what concerns the empirical literature, there is widespread agreement on the fact that firms do have target leverage ratios (usually computed as a linear combination of firm-level variables such as sales, market-to-book ratio and profitability) and that firm leverage follows a partial adjustment dynamics: each year the firm closes part of the gap between current leverage and target one. However, the econometric uncertainties associated with dynamic panel data have made the achievement of a consensus around the magnitude of the adjustment speed very difficult: [11] gives an estimate close to 30% per year for a typical firm, [18] reports one between 17 and 23%, while [8] estimate a speed in the 7–18% range. The reason behind these stark differences may lie behind various data set imperfections, such as missing data, endogeneity and serial correlation, as shown in [10].

Another source of debate is around what determines partial adjustment towards the target: indeed, in a frictionless world, firms would always set their actual leverage to the target one, but adjustment costs may outweigh the gains from removing small
deviations between the two. The search for the sources of adjustment costs has spawned an enormous wealth of research: for instance in [9] the authors find that market conditions and cash flow realizations substantially affect it, while [29] studies the impact of transaction costs on observed leverage patterns. Instead, in the paper [15] the relationship between leverage adjustment and the business cycle is explored: the authors find that leverage dynamics behaves countercyclically once they control for a business-cycle-dependent target.

A possible way to identify the constituents of adjustment costs would be to resort to dynamic models of capital structure in the presence of financial frictions and check under which conditions the optimal leverage policy has a partial adjustment structure. Indeed, for classic models of pure investment/capital like the Solow growth model [27], it is well-known that a partial adjustment towards the optimum is the policy function, see for instance [1] for a formal proof: extending those ideas to a model of investment decisions may identify the key financial frictions that determine partial adjustment. The recent literature contains plenty of such models. For example, in [16] the authors study a partial equilibrium model that predicts path-dependent leverage targets. The same framework, augmented with features of the asset pricing model in [30], is studied in [21]. The paper [25] studies a model that exhibits constant book leverage but varying market leverage across different book-to-market portfolios. The authors of [17] estimate a structural model with the goal of gauging the effects of external finance constraints on investment. The paper [13] studies the relationship between leverage and stock returns in the presence of market imperfections. Finally, [5] estimates a dynamic capital structure model that generates leverage ratios with slow speed of adjustment.

The main challenge of this approach is that these models are fairly complicated, and a study addressing the analytical properties of the policy function for leverage is not currently available. The importance of such a study would be twofold: First, it would reconcile structural models with the findings of the empirical literature, strengthening their conclusions and revealing potential overlooked aspects. Second, understanding how the policy function of leverage behaves is key to resolve the debate on the whether firm capital structure is stable (as argued in [5, 6]) or not (as claimed in [20]).

In this work we shall present a dynamic model of capital structure and we shall state sufficient conditions for the leverage policy function to exhibit partial adjustment. The model is partly inspired by [16] and features a collateral constraint that imposes a tradeoff between the tax advantage of debt and distress costs, guaranteeing risk-free debt contracts. We first prove a sufficient condition for the constraint to be binding (which is, for instance, satisfied whenever the firm is not expecting to change its next period payout policy and next period production is guaranteed to be sufficiently high): whenever the constraint binds, the first order conditions of the financial manager’s optimization problem impose an inverse relationship between optimal capital and leverage, as well as a recursive relationship between the current state of the firm and next period leverage. By resorting to a contraction argument, we show that such functional form yields the existence of a state-dependent target leverage ratio towards which optimal leverage converges. Remarkably, the firm marginal productivity of capital is deeply linked with the speed of convergence, and our result predicts its magnitude to be around 25%, which is in line with most of the empirical estimates cited above. The result also prescribes a countercyclical behavior of optimal capital and leverage with respect to the business cycle, as in [14, 15]: in bad times, firms should accumulate more capital and decrease their leverage ratio in order to counteract the increasing risk of negative shocks to the cash flow process.
The work is organized as follows: Section 2 introduces formally the notion of partial adjustment in its weak and strong form. Section 3 discusses the state variables (capital and debt) of the model, exploring some useful facts regarding the state space, as well as the underlying financial frictions (tax system, flotation costs and collateral constraints). Section 4 introduces leverage in the model, stating the financial manager’s optimization problem and investigating some properties of the value function. Section 5 is devoted to the proof of the partial adjustment result: the collateral constraint is shown to play a fundamental role since its Lagrange multiplier determines optimal capital and leverage dynamics. Theorem 5.4 links analytic properties of the multiplier with the existence of a (state-dependent) target leverage ratio and the behavior of the policy function. Section 6 presents an interpretation of the results from an economic and financial perspective, while Section 7 concludes the work by summarizing the main findings and addressing possible extensions and future developments.

2 Partial adjustment

A standard partial adjustment model of firm capital structure is specified as follows

$$L_{t+1} - L_t = \lambda (L^* - L_t)$$ (1)

where $L_t$ denotes the actual leverage ratio at time $t$, $L^*$ is the long run target leverage ratio, and $\lambda \in (0, 1)$ is the so called speed of adjustment. The interpretation is that a firm closes $\lambda$ percent (per time period) of the gap between present and target leverage. Usually, target leverage is defined as a linear combination of firm specific characteristics $X$, i.e., for some vector $\beta$ it holds

$$L^* = \beta' X.$$ 

A typical example (see [15]) would be to choose

$$X = \{\text{sales, mtb, profit, tang, capex}\},$$

where sales denotes total sales, mtb stands for market-to-book ratio, profit (resp. tang) is a measure of profitability (resp. tangibility), and capex is capital expenditure.

We shall call condition (1) strong partial adjustment with target $L^*$ and speed of adjustment $\lambda$. We distinguish it from weak partial adjustment with target $L^*$ and speed of adjustment $\lambda$, which reads as follows:

$$\|L^* - L_{t+1}\| \leq (1 - \lambda)\|L^* - L_t\|.$$ (2)

It is not surprising that the following result holds.

**Proposition 2.1.** Strong partial adjustment with target $L^*$ and speed of adjustment $\lambda$ implies weak partial adjustment with target $L^*$ and speed of adjustment $\lambda$.

**Proof.** Adding and subtracting $L^*$ from the right-hand side of (1) we obtain

$$L_{t+1} - L_t = L_{t+1} - L^* + L^* - L_t$$

which yields the identity

$$L^* - L_{t+1} = (L_t - L^*) - \lambda (L_t - L^*)$$

$$= (1 - \lambda)(L_t - L^*).$$

Since $\lambda \in (0, 1)$, taking absolute values on both sides we obtain (2) with equality. \qed
In the following we shall focus on establishing under which conditions weak partial adjustment holds, since (requiring only a notion of distance between leverage ratios) it is easier to handle analytically with metric space techniques. However, notice that if weak partial adjustment with target \( L^* \) holds and in addition

\[ L_t < L^* \Rightarrow L_{t+1} \in [L_t, L^*] \text{ and } L^* < L_t \Rightarrow L_{t+1} \in [L^*, L_t], \]

then we can choose \( \lambda(t) \) to satisfy

\[ L_{t+1} - L_t = \lambda(t)(L^* - L_t) \]

and we obtain strong partial adjustment with (time-varying) speed of adjustment \( \lambda(t) \). Hence, weak partial adjustment plus monotonicity of the policy function \( L_t \) implies strong partial adjustment (with same target leverage but, in principle, different speed of adjustment).

3 The model

In this section we shall introduce in details the tradeoff model of capital structure for which we shall prove a weak partial adjustment result.

We consider a discrete-time infinite-horizon setting where investors are homogeneous and risk neutral. We denote by \( r \) the rate of return on the taxable riskless Treasury bill, by \( k \) single-period capital and by \( p \) the face value of single-period debt. Debt \( p \) may assume negative values, representing the fact that the firm is lending instead of borrowing.

**Assumption 1** (Shock dynamics). We assume that

- the space of profit shocks is the topological space \( \mathcal{Z} = [\underline{z}, \overline{z}] \) endowed with \( \mathcal{B}(\mathcal{Z}) \), the Borel subsets of \( \mathcal{Z} \);
- the transition function \( \mathbb{P} : \mathcal{Z} \times \mathcal{B}(\mathcal{Z}) \to [0, 1] \) is monotone, and satisfies the Markov and the Feller properties;

**Assumption 2** (Operating profit). We assume that

- the space of capital inputs satisfies \( \mathcal{K} \subseteq \mathbb{R}_+ \);
- the operating profit function \( \pi : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}_+ \) is strictly increasing in the second component and satisfies the Inada conditions with respect to the first component, that is
  - \( \pi(0, z) = 0 \) for every \( z \in \mathcal{Z} \);
  - it is twice continuously differentiable in \( k \);
  - it is strictly increasing in \( k \);
  - it is strictly concave in \( k \);
  - \( \lim_{k \to 0} \partial_k \pi(k, z) = +\infty \) for every \( z \in \mathcal{Z} \);
  - \( \lim_{k \to +\infty} \partial_k \pi(k, z) = 0 \) for every \( z \in \mathcal{Z} \).

**Remark 3.1.** As noted in [12], due to the hypotheses on \( \pi \) in Assumption 2, any level of capital above \( \overline{k} \) defined as

\[ \pi(\overline{k}, z) - \delta_{\overline{k}} = 0 \]
is not economically profitable. This implies that $\bar{k}$ is an upper bound for the space of capital inputs. Moreover, too low levels of capital will never be optimal since their marginal benefit would be close to infinity. We shall assume henceforth that

$$\mathcal{K} = [\underline{k}, \bar{k}]$$

for some $0 < \underline{k} < \bar{k}$.

In the following we shall consider a Cobb-Douglas functional form for the production functions, i.e.,

$$\pi(k, z) = Azk^\chi,$$

for some $A > 0$ and $\chi \in (0, 1)$. The quantity $Az$ is called total factor productivity and is divided in a constant component $A$ and a time-varying random component $z$. With such production function, a firm marginal productivity of capital is measured by the three exogenous parameters $A$, $z$ and $\chi$: the higher these parameters, the higher the capability of the firm to transform present capital in future cash flow.

We now introduce the financial fricitions that are at the core of our model. The first one, motivated by empirical evidence (see for instance \cite{2, 26}), is direct equity flotation costs.

**Assumption 3** (Flotation costs). For each dollar of external equity paid into the firm, there is a flotation cost $\xi > 0$.

The second friction introduces a tradeoff between tax advantage of debt (in the form of a tax shield) versus distress costs.

**Assumption 4** (Financing). The firm may borrow and lend at the risk-free rate $r$ before taxes. The lender imposes a collateral constraint requiring that the fire sale value of capital plus pledgeable cash flow be sufficient to pay the loan. The fire sale value of capital is equal to the value of capital multiplied by $s < 1$.

Hence, in the event of a liquidation, capital is sold at the depressed price $s < 1$ (normalizing the price of new capital to 1). In order to state formally the collateral constraint, we must first describe the tax system in which the firm operates.

**Assumption 5** (Corporate taxable income and tax function). We assume that

- corporate taxable income ($y$) is equal to operating profits less economic deprecations (occurring at rate $\delta > 0$), less interest expense, plus interest income, i.e.,

$$y(k, p, z) := \pi(k, z) - \delta k - \frac{r}{1 + r}p;$$

- investors’ are taxed at flat rates of $\tau_i \in (0, 1)$ on interest income and $\tau_d \in (0, 1)$ on corporate distributions;

- the corporate tax function $g : \mathbb{R} \to \mathbb{R}$ has the following form

$$g(y) := \begin{cases} 
\tau_c y & \text{if } y \geq 0, \\
0 & \text{if } y < 0,
\end{cases}$$

with $\tau_i < \tau_c < 1$. 


Remark 3.2. The assumption $\tau_c > \tau_i$ in Assumption 5 ensures bounded savings (i.e., there is a lower bound on debt), since it implies that the discount factor on savings

$$\frac{1}{1 + r(1 - \tau_c)}$$

is greater than that on distribution of excess liquidity

$$\frac{1}{1 + r(1 - \tau_i)}$$

for each level of taxable income $y$. Hence the firm finds more convenient to redistribute excess liquidity to external investors instead of saving it. This implies that the discount factor of the model is given by

$$R := \frac{1}{1 + r(1 - \tau_i)}.$$ 

If we denote by $k'$ and $p'$ end-of-period capital and debt, the collateral constraint reads as follows

$$p' \leq s(1 - \delta)k' + \pi(k', z') - g(y(k', p', z')),$$

where $z'$ denotes the realized end-of-period shock.

By Assumption 4, the validity of (3) ensures that debt is risk-free (since there is always enough capital to back it up), and hence its rate of return is $r$ (the same as a Treasury bill).

Remark 3.3. We now show that the only collateral constraint that will eventually be binding is the one associated with the worst productivity shock $z'$. Indeed, denote by

$$H(k, p, z) := \pi(k, z) - g(y(k, p, z))$$

and assume that $z < z'$. We want to show that

$$H(k, p, z) < H(k, p, z'),$$

which is equivalent to

$$\pi(k, z) - g(y(k, p, z)) < \pi(k, z') - g(y(k, p, z')).$$

Since $y(k, p, z)$ is strictly increasing in $z$, it holds $g(y(k, p, z)) \leq g(y(k, p, z'))$. By the functional form of $g$, there are only three possibilities: either

$$g(y(k, p, z)) = g(y(k, p, z')) = 0,$$

or

$$g(y(k, p, z)) = \tau_c y(k, p, z) \quad \text{and} \quad g(y(k, p, z')) = \tau_c y(k, p, z'),$$

or

$$g(y(k, p, z)) = 0 \quad \text{and} \quad g(y(k, p, z')) = \tau_c y(k, p, z').$$

In the first and second cases, the thesis follows from the strict increasing monotonicty of $\pi$. In the last case, since $g(y(k, p, z)) = 0$ it follows that $y(k, p, z) < 0$, which can be equivalently rewritten as

$$\pi(k, z) < \delta k + \frac{r}{1 + r} p.$$  (4)
Therefore, the inequality to be proved becomes
\[ \pi(k, z) < \pi(k, z') - \tau_c \pi(k, z') + \tau_c \delta k + \frac{r}{1 + r} p. \]

However notice that, from the inequality \(4\), if we prove the following inequality
\[ \pi(k, z) < \pi(k, z') - \tau_c \pi(k, z') + \tau_c \pi(k, z) \]
we are done. But this follows from the assumption that \(\tau_c < 1\).

**Remark 3.4.** The collateral constraint \(5\) implies the existence of an upper bound for the space of feasible debt states \(P\), provided that the condition
\[ 1 + r(1 - \tau_c) \geq 0 \quad (5) \]
holds. To show it, notice that for every \(k \in K\), the collateral constraint implies either
\[ p \leq s(1 - \delta)k + \pi(k, z) \]
or
\[ p \leq s(1 - \delta)k + \pi(k, z) - \tau_c \pi(k, z) + \tau_c \delta k + \frac{r}{1 + r} p, \]
depending on the sign of \(y(k, p, z)\). In the second case we may rewrite the inequality as
\[ \left(1 - \tau_c \frac{r}{1 + r}\right) p \leq (1 - \delta)k + (1 - \tau_c)\pi(k, z) + \tau_c \delta k, \]
which, if \(5\) holds, yields the upper bound
\[ p \leq \frac{1 + r}{1 + r(1 - \tau_c)} \left( (s(1 - \delta)k + \tau_c \delta k) + (1 - \tau_c)\pi(k, z) \right). \]

Therefore, since \(k\) belongs to the compact set \(K\), and both upper bounds are continuous in \(k\), they both attain their maximum over \(K\): the greatest of the two is the desired upper bound for the space of feasible debt states. Combining this result with Remark \(3\), we infer that
\[ P = [p, \bar{p}] \]
for some finite \(p < \bar{p}\).

Once the financial manager has decided the levels of next period capital \(k_{t+1}\) and debt \(p_{t+1}\) given \(k_t \) and \(p_t\), he shall distribute an amount of cash to shareholders (before taxes and flotation costs) equal to operating profits less corporate taxes less interest payments. If such quantity is positive, it takes the form of a dividend payment, subject to the marginal tax rate \(\tau_d\); if instead it is negative, it is interpreted as an equity issuance, which incurs in the flotation costs of Assumption \(3\).

**Definition 3.5 (Net cash flow to shareholders).** The cash flow to shareholders is defined as
\[ c(k, p, k', p', z) := \pi(k, z) - g(y(k, p, z)) - p - k' + (1 - \delta)k + \frac{p'}{1 + r}. \]

Moreover, let
\[ \Psi(k, L, k', L', z) := \begin{cases} 1 + \xi & \text{if } c(k, L, k', L', z) \geq 0, \\ 1 - \tau_d & \text{if } c(k, L, k', L', z) < 0. \end{cases} \]

Then the *net cash flow to shareholder* is given by
\[ e(k, p, k', p', z) := \Psi(k, L, k', L', z)c(k, p, k', p', z). \]
Remark 3.6. The function $e$ is continuous, and furthermore it is piecewise linear in $p$ and $p'$. Moreover it is differentiable whenever

$$c(k, p, k', p', z) \neq 0 \quad \text{and} \quad y(k, p, z) \neq 0,$$

which are conditions that are usually met in real scenarios.

We are finally able to state formally the financial manager’s optimization problem: maximize the discounted value of net cash flow to shareholders. Given an initial capital structure $(k_0, p_0)$ the financial manager should solve

$$\max_{(k_t, p_t)} \mathbb{E} \left[ \sum_{t=0}^{\infty} R^t e(k_t, p_t, k_{t+1}, p_{t+1}, z_t) \right]$$

subject to the collateral constraint

$$p_{t+1} \leq s(1 - \delta)k_{t+1} + \pi(k_{t+1}, z) - g(y(k_{t+1}, p_{t+1}, z)),$$

and where $k_t \in K$ and $p_t \in P$ for each $t \geq 0$.

Remark 3.7 (Nontrivial debt condition). Since the objective functional of the optimization problem is piecewise linear in $p_{t+1}$, we can easily compute the marginal benefit of debt assuming we are at a differentiability point of both $e(k_t, p_t, k_{t+1}, p_{t+1}, z_t)$ and $e(k_{t+1}, p_{t+1}, k_{t+2}, p_{t+2}, z_{t+1})$. Such quantity is equal to the partial derivative with respect to $p_{t+1}$ of the Lagrangian associated to the problem $\mathcal{L}$ that is

$$\partial_{p_{t+1}} \mathcal{L} = \frac{\Psi_t}{1 + r} - RE_{z_{t+1}} \left[ \Psi_{t+1} \frac{1 + r(1 - g'(y(k_{t+1}, p_{t+1}, z_{t+1})))}{1 + r} \right].$$

Since, in a sufficiently small neighborhood of a differentiability point of the function $e$, the function $g'$ is either equal to 0 or to $c$, the above first order condition implies that we have a corner solution for $p_{t+1}$ depending on the sign of the quantity

$$\Delta_t := \frac{\Psi_t}{1 + r} - RE_{z_{t+1}} \left[ \Psi_{t+1} \frac{1 + r(1 - g'(y(k_{t+1}, p_{t+1}, z_{t+1})))}{1 + r} \right].$$

Indeed, notice that whenever the negative sign holds, the optimal choice of debt is given by the lower bound $p$, hence the above optimization problem becomes a pure capital problem. Instead, if $\Delta_t > 0$, then the optimal strategy is to take as much debt as possible, that is the collateral constraint (2) is binding. Therefore, we say that the nontrivial debt condition holds if

$$\Delta_t > 0.$$

We now show that the nontrivial debt condition is satisfied whenever the financial manager is not expecting to change period $t$ payout policy in period $t + 1$ and the probability he assigns to $y$ being positive in period $t + 1$ is sufficiently high. In this case it holds $\Psi_t = \Psi_{t+1}$, and if we denote by $\nu$ the probability the manager assigns to $y$ being positive we may rewrite $\Delta_t$ as

$$\Delta_t = \frac{\Psi_t}{1 + r} \left( 1 - \frac{\nu(1 + r(1 - \tau_c)) + (1 - \nu)(1 + r)}{1 + r(1 - \tau_i)} \right)$$

$$= \frac{\Psi_t}{1 + r} \left( 1 + r(1 - \tau_i) - \nu(1 + r(1 - \tau_c)) - (1 - \nu)(1 + r) \right)$$

$$= \frac{\Psi_t}{1 + r} \left( 1 + r(1 - \tau_i) - 1 - r(1 - \nu \tau_c) \right)$$

$$= \frac{\Psi_t}{1 + r} \left( 1 + r(1 - \tau_i) - 1 - r(1 - \nu \tau_c) \right).$$
\[= \frac{\Psi_t}{1 + r} \left( \frac{\tau(\nu \tau_c - \tau_i)}{1 + r (1 - \tau_i)} \right).\]

The above quantity is positive whenever
\[\tau_i < \nu \tau_c,\]
which is trivially satisfied, for instance, if \(\nu = 1\).

### 4 Introducing leverage

As already seen in Section 2, partial adjustment results are formulated in terms of leverage, which in the current model can be defined as outstanding debt over capital, that is
\[L_t := \frac{p_t}{k_t}.\]

We may reformulate the model introduced in the previous section in terms of capital \(k\) and leverage \(L\) by multiplying and dividing by \(k\) each occurrence of \(p\). For instance, corporate taxable income can be rewritten as
\[
\tilde{y}(k, L, z) := y(k, kL, z) = \pi(k, z) - \frac{r}{1 + r} kL,
\]
and the new collateral constraint becomes
\[k' L' \leq s (1 - \delta) k' + \pi(k', z) - g(\tilde{y}(k', L', z)).\]

Under the previous assumptions on the tax function \(g\), the collateral constraint can be rewritten as
\[C(k', L', z) \geq 0\]
where
\[C(k, L, z) := (1 - \delta) + g(\tilde{y}(k, L, z)) \delta + (1 - g(\tilde{y}(k, L, z))) \frac{\pi(k, z)}{k} - \frac{1 + r (1 - \tau_c)}{1 + r} L.\]

Notice that we divided both sides of the collateral constraint by \(k\), which is allowed since we already argued that it is strictly positive.

In what follows, denote by \(g_t := g(\tilde{y}(k_t, L_t, z_t))\) and assume that \(\tilde{y}(k_t, L_t, z_t) \neq 0\). The following preliminary computations of the partial derivatives of the collateral constraint shall be useful when we will derive the first order conditions of the optimization problem:

\[
\alpha(k_t, z_t) := \partial_{k_t} C(k_t, L_t, z_t) = (1 - g_t') \partial_{k_t} \frac{\pi(k_t, z_t)}{k},
\]
\[
\beta_t(z_t) := \partial_{L_t} C(k_t, L_t, z_t) = - \frac{1 + r (1 - \tau_c) L}{1 + r},
\]
where \(g_t'\) is either equal to 0 or to \(\tau_c\) (depending on the sign of \(\tilde{y}(k_t, L_t, z_t)\)).

**Remark 4.1.** In both the expressions of \(\alpha\) and \(\beta\), leverage does not appear. Moreover, the nontrivial debt condition can be rewritten using \(\beta_t\) as
\[\Delta_t = \frac{\Psi_t}{1 + r} + RE_{z_t+1} [\Psi_{t+1} \beta_{t+1}(z_{t+1}) | z_t] > 0.\]
Remark 4.2. Concerning the set $\mathcal{K}$, nothing as changed between the current framework and that of Section 3 since both bounds on $\mathcal{K}$ are determined by the properties of $\pi$. Regarding the bounds of the set of feasible leverage states $\Lambda$, the existence of a lower bound follows again from Remark 3.2. The existence of an upper bound, instead, follows from the collateral constraint, which may be rewritten for the purpose as

\[ L \leq \frac{1 + \delta (1 - g_t)}{1 + r (1 - g_t)} (1 + r) + \frac{(1 + r)(1 - g_t) \pi(k, z)}{1 + r (1 - g_t) k}. \]

From this, since the function $\pi(k, z)/k = Azk^{-1}$ is continuous on the compact set $\mathcal{K}$ (as $0 \notin \mathcal{K}$), it attains its maximum, which implies the existence of an upper bound. Therefore $\Lambda$ is a compact set.

The functional form of $g$ allows us to rewrite the cash flow to shareholders function in terms of capital and leverage as follows:

\[ c(k, L, k', L', z) := (1 - g_t)\pi(k, z) - \frac{1 + r (1 - g_t)}{1 + r} kL - k' + (1 - \delta + g_t \delta) k + \frac{k'L'}{1 + r}. \]

The net cash flow to shareholders is then defined as

\[ c(k, L, k', L', z) := \Psi(k, L, k', L', z)c(k, L, k', L', z). \]

Like in Remark 3.6, the function $e$ is not differentiable whenever

\[ c(k_t, L_t, k_{t+1}, L_{t+1}, z) = 0 \quad \text{or} \quad \tilde{y}(k_t, L_t, z) = 0. \]

Outside of these cases, its partial derivatives are equal to

\[
\begin{align*}
\partial_{k_t}c(k_t, L_t, k_{t+1}, L_{t+1}, z) &= \Psi_t \left( (1 - g'_t)\partial_{k_t}\pi(k_t, z) + (1 - \delta + g'_t \delta) - \frac{1 + r(1 - g'_t)}{1 + r} L_t \right) \\
&= \gamma(k_t, z_t) + \Psi_t \beta_t(z_t)L_t, \\
\partial_{L_t}c(k_t, L_t, k_{t+1}, L_{t+1}, z) &= -\Psi_t \frac{1 + r(1 - g'_t)}{1 + r} k_t \\
&= \Psi_t \beta_t(z_t)k_t, \\
\partial_{k_{t+1}}c(k_t, L_t, k_{t+1}, L_{t+1}, z) &= -\Psi_t + \frac{\Psi_t}{1 + r} L_{t+1}, \\
\partial_{L_{t+1}}c(k_t, L_t, k_{t+1}, L_{t+1}, z) &= \frac{\Psi_t}{1 + r} k_{t+1},
\end{align*}
\]

where we denoted by $\gamma(k_t, z_t) := (1 - g'_t)\partial_{k_t}\pi(k_t, z_t) + (1 - \delta + g'_t \delta)$.

Remark 4.3. Under the assumption $\pi(k, z) = Azk^{-1}$, we may compute the functions $\alpha$ and $\gamma$ explicitly. Indeed we have

\[ \alpha(k_t, z_t) = (1 - g'_t)\partial_{k_t}\pi(k_t, z_t) = (1 - g'_t)(\chi - 1)Az_tk_t^{-2} \]

as well as

\[ \gamma(k_t, z_t) = (1 - g'_t)\partial_{k_t}\pi(k_t, z_t) + (1 - \delta + g'_t \delta) = (1 - g'_t)\chi Az_tk_t^{-1} + (1 - \delta + g'_t \delta). \]
The financial manager’s optimization problem in the presence of leverage becomes: given an initial capital structure \((k_0, L_0)\), solve

\[
\max_{(k_t, L_t) \in \mathcal{K} \times \mathcal{L}} \mathbb{E} \left[ \sum_{t=0}^{\infty} R^t e(k_t, L_t, k_{t+1}, L_{t+1}, z_t) \right]
\]

subject to \(C(k_{t+1}, L_{t+1}, z) \geq 0\),

where \(k_t \in \mathcal{K}\) and \(L_t \in \mathcal{L}\) for each \(t \geq 0\).

The Bellman equation associated to this problem is

\[
V(k, L, z) = \max_{(k', L') \in \Gamma} e(k, L, k', L', z) + R \int_{\mathcal{Z}} V(k', L', z') P(z, dz')
\]

where the constraint set \(\Gamma\) is defined as

\[
\Gamma = \{(k, L) \in \mathcal{K} \times \mathcal{L} \mid C(k, L, z) \geq 0\}.
\]

**Theorem 4.4.** There is a unique continuous function \(V : \mathcal{K} \times \mathcal{L} \times \mathcal{Z} \to \mathbb{R}\) satisfying (9).

**Proof.** We simply need to show that the operator \(M : C(\mathcal{K} \times \mathcal{L} \times \mathcal{Z}) \to C(\mathcal{K} \times \mathcal{L} \times \mathcal{Z})\) defined as

\[
(Mf)(k, L, z) = \max_{(k', L') \in \Gamma} e(k, L, k', L', z) + R \int_{\mathcal{Z}} f(k', L', z') P(z, dz')
\]

is a contraction with modulus \(R\). But this follows from Theorem 9.6 of [28] since the Assumptions 9.4–9.7 of the same reference are satisfied. In particular:

- the domain \(\mathcal{K} \times \mathcal{L}\) is convex and Borel;
- the set \(\mathcal{Z}\) is compact and the transition probability \(P\) has the Feller property;
- the constraint set \(\Gamma\) is constant-valued, hence compact-valued and continuous;
- the function \(e\) is bounded and continuous, and \(R \in (0, 1)\).

Moreover, since both \(e\) and \(f\) are continuous and \(\Gamma\) is compact, then the supremum is attained. \(\Box\)

## 5 The main result

In this section, we will compute the first-order conditions of the optimization problem (8) in order to obtain a partial adjustment result. The Lagrangian associated to the problem is given by

\[
\mathcal{L} := \mathbb{E} \left[ \sum_{t=0}^{\infty} R^t \left( e(k_t, L_t, k_{t+1}, L_{t+1}, z_t) + P(z_t, \overline{z}) C(k_{t+1}, L_{t+1}, \overline{z}) \mu(L_t, k_t, z_t) \right) \right],
\]

where \(\mu(L_t, k_t, z_t)\) denotes the Lagrange multiplier of the collateral constraint for the shock \(\overline{z}\). Notice that the constraint will enter into play only if at \(t+1\) the shock is \(\overline{z}\), since by Remark 3.3 the Lagrange multiplier for a shock different from \(\overline{z}\) is always equal to 0. This explains why this term is multiplied by the transition probability from the current shock \(z_t\) to \(\overline{z}\).
For the rest of the section we assume that the nontrivial debt condition holds at \( t \) (i.e., \( \Delta_t > 0 \)), as well as
\[
ed(k_t, L_t, k_{t+1}, L_{t+1}, z) \neq 0 \quad \text{and} \quad \tilde{y}(k_t, L_t, z) \neq 0
\]
and
\[
ed(k_{t+1}, L_{t+1}, k_{t+2}, L_{t+2}, z) \neq 0 \quad \text{and} \quad \tilde{y}(k_{t+1}, L_{t+1}, z) \neq 0.
\]
Under these assumptions, the partial derivatives of \( L \) with respect to next-period capital and leverage are well defined and we can compute
\[
\partial_{k_t+1} L = R_t^e \partial_{k_t+1} e(k_t, L_t, k_{t+1}, L_{t+1}, z_t) +
R_t^e \mathbb{E}(z_{t+1}) \partial_{k_t+1} C(k_{t+1}, L_{t+1}, z_{t+1}) \mu(L_t, k_t, z_t) +
R_t^{t+1} \mathbb{E}_{z_{t+1}} \left[ \partial_{k_t+1} e(k_{t+1}, L_{t+1}, k_{t+2}, L_{t+2}, z_{t+1}) \mid z_t \right] = 0,
\]
as well as
\[
\partial_{L_t+1} L = R_t^e \partial_{L_t+1} e(k_t, L_t, k_{t+1}, L_{t+1}, z_t) +
R_t^e \mathbb{E}(z_{t+1}) \partial_{L_t+1} C(k_{t+1}, L_{t+1}, z_{t+1}) \mu(L_t, k_t, z_t) +
R_t^{t+1} \mathbb{E}_{z_{t+1}} \left[ \partial_{L_t+1} e(k_{t+1}, L_{t+1}, k_{t+2}, L_{t+2}, z_{t+1}) \mid z_t \right] = 0.
\]

Using the first order condition with respect to \( L_{t+1} \) (the second chain of identities) we obtain
\[
\Psi_t \frac{\beta_{t+1}}{1 + r} k_{t+1} + \beta_{t+1} \mathbb{E}(z_{t+1}) \mu(L_t, k_t, z_t) + R_t^{t+1} \mathbb{E}_{z_{t+1}} \left[ \Psi_t \beta_{t+1} (z_{t+1}) \mid z_t \right] k_{t+1} = 0,
\]
which can be rearranged as follows
\[
\left( \Psi_t \frac{1}{1 + r} + R_t^{t+1} \mathbb{E}_{z_{t+1}} \left[ \Psi_t \beta_{t+1} (z_{t+1}) \mid z_t \right] \right) k_{t+1} = -\beta_{t+1} \mathbb{E}(z_{t+1}) \mu(L_t, k_t, z_t).
\]
The expression on the RHS is equal to \( \Delta_t k_{t+1} \), hence we get
\[
k_{t+1} = -\beta_{t+1} \mathbb{E}(z_{t+1}) \mu(L_t, k_t, z_t) \frac{1}{\Delta_t}.
\]

**Remark 5.1.** To keep the notation compact, we shall denote by
\[
\phi_t := -\beta_{t+1} \mathbb{E}(z_{t+1}) \mu(L_t, k_t, z_t) \frac{1}{\Delta_t},
\]
which let us rewrite (10) as \( k_{t+1} = \phi_t \mu(L_t, k_t, z_t) \). This identity has several nontrivial implications:

- first, notice that \( \beta_{t+1} (z) \) is defined as a negative quantity and that the multiplier \( \mu \) is always nonnegative, hence \( k_{t+1} \) is nonnegative (coherently with the definition of the set of capital inputs \( \mathcal{K} \));
- since \( \mathcal{K} \) is a subset of \( \mathbb{R}_+ \) that does not contain 0, and \( k_{t+1} \in \mathcal{K} \), it follows that \( \mu(L_t, k_t, z_t) > 0 \), that is the collateral constraint of the shock \( z \) is always binding. Notice that this holds because we are assuming that \( \Delta_t > 0 \);
more precisely, since $\mathcal{K} = [\underline{k}, \overline{k}]$ it follows

$$\mu(L_t, k_t, z_t) \in \left[ \frac{k}{\phi_t}, \frac{\overline{k}}{\phi_t} \right];$$  \hspace{1cm} (11)

the smaller $\mathbb{P}(z_t, \underline{z})$, the lower the amount of optimal capital $k_{t+1}$. An economic interpretation is the following: if $\mathbb{P}(z_t, \underline{z})$ decreases then next-period expected productivity of capital (for a given level of capital) increases, therefore the amount of capital $k_{t+1}$ the firm needs to reach optimal output decreases.

To obtain the optimal level of leverage $L_{t+1}$ we use the first order condition with respect to $k_{t+1}$, that reads as follows:

$$-\Psi_t + \frac{\Psi_t}{1 + r} \Delta t L_{t+1} + \alpha(k_{t+1}, \underline{z})\mathbb{P}(z_t, \underline{z}) \mu(L_t, k_t, z_t) + R\mathbb{E}_{z_t+1}[\gamma(k_{t+1}, z_{t+1}) | z_t] +$$

$$+ R\mathbb{E}_{z_t+1}[\Psi_{t+1} \beta_{t+1}(z_{t+1}) | z_t] L_{t+1} = 0.$$  

Rearranging we get

$$\Delta_t L_{t+1} = \Psi_t - \alpha(k_{t+1}, \underline{z})\mathbb{P}(z_t, \underline{z}) \mu(L_t, k_t, z_t) - R\mathbb{E}_{z_t+1}[\gamma(k_{t+1}, z_{t+1}) | z_t].$$

Using the identity $k_{t+1} = \phi_t \mu(L_t, k_t, z_t)$ and denoting by $\mu = \mu(L_t, k_t, z_t)$ we obtain

$$\Delta_t L_{t+1} = \Psi_t - \alpha(\phi_t \mu, \underline{z})\mathbb{P}(z_t, \underline{z}) \mu - R\mathbb{E}_{z_t+1}[\gamma(\phi_t \mu, z_{t+1}) | z_t].$$

By Remark [4.3] it follows that

$$L_{t+1} = \omega_1(z_t) \mu(L_t, k_t, z_t)^{\chi-1} + \omega_2,$$  \hspace{1cm} (12)

where

$$\omega_1(z_t) := \frac{A}{\Delta_t \phi_t^{2-\chi}} \left( (1 - g_t)(1 - \chi)z_t \mathbb{P}(z_t, \underline{z}) - \chi R\mathbb{E}_{z_t+1}[(1 - g_{t+1}) z_{t+1} | z_t] \right)$$

$$\omega_2 := \frac{\Psi_t + (1 - \delta + g_t \delta)}{\Delta_t}.$$  

**Remark 5.2.** Since we know that $k_{t+1} = \phi_t \mu(L_t, k_t, z_t)$, the above relationship implies that

$$L_{t+1} = \frac{\omega_1(z_t) \phi_t^{1-\chi}}{k_{t+1}^{\chi}} + \omega_2,$$  \hspace{1cm} (14)

that is, optimal leverage is inversely proportional to optimal capital.

**Remark 5.3.** If $\mu(\cdot, k_t, z_t)$ is continuous with respect to $L$, from the condition

$$\mu(L_t, k_t, z_t) \geq \frac{k}{\phi_t} > 0,$$

it follows that there exists a neighborhood of $L_t$, which we denote by $\mathcal{B}(L_t, k_t, z_t)$, inside which the quantity

$$m_{\mathcal{B}(L_t, k_t, z_t)} := \inf_{L \in \mathcal{B}(L_t, k_t, z_t)} |\mu(L, k_t, z_t)|$$

is strictly positive.
In the following, we shall assume that the function $e$ is differentiable inside $B(L_t, k_t, z_t)$. We are now ready to state our main result.

**Theorem 5.4.** Assume that the Lagrange multiplier function $\mu(\cdot, k_t, z_t)$ is continuously differentiable in $B(L_t, k_t, z_t)$ with local Lipschitz constant satisfying

$$\text{Lip}(\mu(\cdot, k_t, z_t); B(L_t, k_t, z_t)) \leq \frac{m_2 - \chi}{m_{B(L_t, k_t, z_t)}}. \quad (15)$$

Then there exists an $L^*(L_t, k_t, z_t) \in B(L_t, k_t, z_t)$ such that leverage follows a weak partial adjustment dynamics with target leverage $L^*(L_t, k_t, z_t)$ and speed of adjustment $\chi$.

**Proof.** Denote by

$$h(L, k, z) := \omega_1(z) \mu(L, k, z)^{\chi-1} + \omega_2.$$

Then, by assumption, the function $h(\cdot, k_t, z_t)$ is locally continuously differentiable inside $B(L_t, k_t, z_t)$, and from (12) it satisfies

$$L_{t+1} = h(L_t, k_t, z_t).$$

Moreover, by (11) its derivative at $L \in B(L_t, k_t, z_t)$ satisfies

$$|D_L h(L, k_t, z_t)| = |\omega_1(z_t) D_L \mu(L, k_t, z_t)^{\chi-1}|$$

$$= \left| \frac{(\chi - 1) \omega_1(z_t)}{\mu(L, k_t, z_t)^{2-\chi}} \right| \left| D_L \mu(L, k_t, z_t) \right|$$

$$\leq (1 - \chi) \left| \frac{\omega_1(z_t)}{m_{B(L_t, k_t, z_t)}} \right| \left| D_L \mu(L, k_t, z_t) \right|$$

$$\leq (1 - \chi) \left| \frac{\omega_1(z_t)}{m_{B(L_t, k_t, z_t)}} \right| \text{Lip}(\mu(\cdot, k_t, z_t); B(L_t, k_t, z_t))$$

$$\leq 1 - \chi.$$

Hence, for every $L_1, L_2 \in B(L_t, k_t, z_t)$ by Lagrange Theorem it holds

$$\|h(L_1, k_t, z_t) - h(L_2, k_t, z_t)\| \leq \sup_{L \in B(L_t, k_t, z_t)} |D_L h(L, k_t, z_t)||L_1 - L_2|$$

$$\leq (1 - \chi) \|L_1 - L_2\|.$$

The above inequality implies that, inside $B(L_t, k_t, z_t)$, the function $h(\cdot, k_t, z_t)$ is a contraction with modulus $1 - \chi$, and therefore by the Contraction Mapping Theorem there exists an $L^*(L_t, k_t, z_t) \in B(L_t, k_t, z_t)$ that satisfies

$$L^*(L_t, k_t, z_t) = h(L^*(L_t, k_t, z_t), k_t, z_t).$$

Notice that $L^*(L_t, k_t, z_t)$ depends on $L_t, k_t$ and $z_t$ via $h$ and $B$. By a well-known property of contraction mappings, the function $h(\cdot, k_t, z_t)$ satisfies

$$\|h(L_t, k_t, z_t) - L^*(L_t, k_t, z_t)\| \leq (1 - \chi) \|L_t - L^*(L_t, k_t, z_t)\|,$$

and by replacing $L_{t+1} = h(L_t, k_t, z_t)$ we obtain the definition of weak partial adjustment with target leverage $L^*(L_t, k_t, z_t)$ and speed of adjustment $\chi$. $\square$
6 Interpretation and extensions

We shall now give an economic and financial interpretation to the results we obtained in the previous sections, and analyze how the various components of the model affect partial adjustment. We shall also discuss how our findings fit with the evidences of the empirical literature, and try to propose possible extensions of our results.

6.1 Financial frictions and partial adjustment

Looking back at the argument leading to Theorem 5.4, we see how the nontrivial debt condition plays a prominent role in the argument. Whenever it holds, the benefits of debt exceed its costs and it is convenient for firms to be as highly levered as possible. However, in the spirit of the tradeoff theory of capital structure, such benefits are capped by the ability of the firm to pledge its capital and cash flow in order to avoid distress costs.

Our result shows how such market imperfection influences the financial manager’s decisions: when the collateral constraint binds, the Lagrange multiplier of the constraint becomes the driver of the dynamics of both capital and leverage policy functions, and its sensitivity to changes to current leverage ratios gives a sufficient condition for partial adjustment to hold.

6.2 Speed of adjustment and output elasticity of capital

Under the assumptions of Theorem 5.4, leverage follows a partial adjustment dynamics with speed of adjustment given by $\chi$, the output elasticity of capital. If we were to use a value of $\chi$ around 0.25, as originally estimated by Cobb and Douglas in [4] and later confirmed by the National Bureau of Economic Research, this result predicts a speed of adjustment of 25%, which is in line with most of the results obtained by the empirical literature: just to mention a few in addition to those already reported in the introduction, the authors in [9] obtain an estimate in the 20-30% bracket, while those in [20] get an estimate of exactly 25%.

More generally, our result links the capability of a firm to convert capital into cash flow with how fast it converges to the target leverage ratio. It predicts that the closer $\chi$ is to 1, the faster current leverage converges to the target, since for a value of $\chi \approx 1$, the relationship (2) becomes

$$\|L_{t+1} - L^*\| \approx 0,$$

which implies $L_{t+1} \approx L^*$, that is the target is (almost) reached in one period. We shall discuss more in depth the relationship between partial adjustment and the productivity of capital in the next section.

6.3 Partial adjustment, productivity of capital and the business cycle

Firm marginal productivity of capital is deeply related with the right-hand side of (15), and in particular with the term $|\omega_1(z_t)|$: by definition, the function $\omega_1$ is directly proportional to the quantity $A$ and inversely proportional to the quantity $\phi_t^{2-\chi}$. This implies that

$$A \downarrow \text{ or } \chi \downarrow \Rightarrow |\omega_1(z_t)| \downarrow.$$
Moreover, the relationship between $A$ and the productivity of capital is quite clear: the lower this value, the lower the total factor productivity of the production function, hence the lower next period expected profit (for a given level of capital). Similarly, a decrease in $\chi$ reduces the output elasticity of capital. If the marginal productivity of capital decreases, next period optimal level of capital will decrease (since its marginal benefit is lower). Optimal capital and optimal leverage are linked by the relationship (14), so if the first decreases then the second increases: this implies that the productivity of capital influences inversely the size of optimal leverage. This analysis shows that a decrease in marginal productivity of capital increases the right-hand side of (15) (so the partial adjustment result is more likely to hold) as well as next period leverage ratio.

An economic interpretation of this phenomenon could be the following: when the marginal productivity of capital is low, the firm finds optimal to finance its operations through channels that are alternative to production, because of its reduced capability in transforming capital into cash flow. In such context, debt (and, consequently, leverage) plays a prominent role in the financial manager’s optimal schedule, and leverage is more likely to follow a partial adjustment dynamics (which we have shown to be the policy function). Instead, when the productivity of capital is very high, then the firm is quite efficient on the production side and the role of leverage becomes more marginal. When this happens, our result “breaks down” as for very low levels of leverage it might be that the policy function is much more unstable than a partial adjustment one.

However, a natural question arises at this point: since it is well-known that for structural models of pure investment, the policy function for capital has a partial adjustment form, see [1], is it also true for our model when capital dominates over leverage? If this holds, we would have two regimes: in the low-productivity one, we would have partial adjustment in the leverage policy function, while in the high-productivity one it is optimal capital that follows a partial adjustment dynamics.

Partial adjustment is also affected by business cycle fluctuations via the quantity $P(z_t, \bar{z})$ (the probability to be hit by the worst possible shock). In fact, during recessions, $P(z_t, \bar{z})$ is likely to increase, which leads ceteris paribus to an increase in $k_{t+1}$ via (10). But then, optimal leverage decreases following the relationship (14), which implies that in bad times firms should seek less aggressive leverage schedules. This is in line with the empirical results of [14], which claims that leverage ratios are larger during expansions than during recessions.

6.4 The behavior of target leverage

Since target leverage is obtained by means of a nonconstructive argument in the proof of Theorem 6.4, our result is essentially mute on how $L^*$ depends on firm state variables (which in our model are $L_t, k_t$ and $z_t$). However, since $L^*$ is the basin of attraction of the function $h(\cdot, k_t, z_t)$ inside the set $B(L_t, k_t, z_t)$, it is likely to depend on how these two objects depend on the state variables $(L_t, k_t, z_t)$: for instance, if they are rather insensitive to changes in $L_t, k_t$ and $z_t$, then target leverage is likely to remain quite stable.

The above argument shows that our result is not able to settle the debate around the stability of firm capital structure (the view of [5, 6] versus the thesis of [20]). Nonetheless, it suggests that the answer to such question may be found by inspecting more closely the properties of the Lagrange multiplier $\mu(L_t, k_t, z_t)$, which determines both $h(\cdot, k_t, z_t)$ and $B(L_t, k_t, z_t)$, and in particular how sensible it is to changes to beginning-of-period state variables.
6.5 Conditions on the Lagrange multiplier

The assumptions of Theorem 5.4 are stated in terms of the Langrange multiplier \( \mu \) rather than in terms of the model’s components like the functions \( \pi, g \), etc. A natural question is then: under which conditions on the primitives of the model the assumptions of Theorem 5.4 are satisfied? We now propose a possible strategy to answer this question.

Under the non-trivial debt condition and the differentiability of the function \( e \), we showed in Section 5 that the policy function (if it exists) satisfies

\[
G(L_t, k_t, z_t) = (\phi_1 \mu(L_t, k_t, z_t), \omega_1(z_t) \mu(L_t, k_t, z_t)^\chi - 1 + \omega_2),
\]

where the first (resp. second) component of \( G \) is optimal capital (resp.) leverage at \( t + 1 \). This implies that the policy function \( G \) is a nonlinear smooth function of the Lagrange multiplier: therefore, if we were able to show that the policy function is sufficiently smooth, this would imply that also the Lagrange multiplier is.

Results about the differentiability of the policy function are not many, as the topic is quite difficult and it is usually necessary to resort to ad hoc model-related assumptions, see for instance [3, 23]. The major drawback of these results is that they require the concavity of the function \( e \) with respect to \( k \) and \( L \): in our case, unfortunately, \( e \) is not concave because of the cross terms \( kL \). If we were able to replace those terms with a concave counterpart, those results could become available and would allow us to translate the requests on the multiplier \( \mu \) in assumptions on the function \( e \), and thus on the various market imperfections that determine it.

7 Conclusions

In this work we showed how financial and economic frictions are able to generate a partial adjustment dynamics in leverage policy functions. In the model we studied, the key factors of this phenomenon are collateral constraints (which strike a balance between tax benefits of debt and distress costs) and firm productivity of capital. The latter, in particular, determines the speed of adjustment towards the (state-dependent) target leverage ratio.

Our model fits well several stylized facts of leverage dynamics established by the empirical literature: an example is given by the magnitude of the speed of adjustment, which falls into the confidence intervals estimated by several authors. Another one, is the countercyclical behavior of leverage dynamics with respect to the business cycle, which is due to the fact that in recessions it is easier for the collateral constraint to be binding.

Future work should first address the translation of the hypotheses of Theorem 5.4 on the Lagrange multiplier into assumptions on the components of the model (the production function and the various market frictions). The next step would then be to extend the model to a full general equilibrium model to study thoroughly the effects of preference and monetary shocks on leverage dynamics. Pairing consumers’ utility maximization with firms’ financing problem would also allow to study the interaction between expected returns and partial adjustment: in such framework, the collateral constraint should probably be replaced by several credit rating inequalities determining both firm specific discount rates and target leverage ratios.

References


