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Consistency of heterogeneous synchronization patterns in complex weighted networks

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Synchronization within the dynamical nodes of a complex network is usually considered homogeneous through all the nodes. Here we show, in contrast, that subsets of interacting oscillators may synchronize in different ways within a single network. This diversity of synchronization patterns is promoted by increasing the heterogeneous distribution of coupling weights and/or asymmetries in small networks. We also analyze consistency, defined as the persistence of coexistent synchronization patterns regardless of the initial conditions. Our results show that complex weighted networks display richer consistency than regular networks, suggesting why certain functional network topologies are often constructed when experimental data are analyzed. © 2017 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). [http://dx.doi.org/10.1063/1.4977972]

Dynamical systems may synchronize in several ways, at the same time, when they are coupled in a single complex network. Examples of this diversity of synchronization patterns may be found in research fields as diverse as neuroscience, climate networks, or ecosystems. Here we report the conditions required to obtain coexisting synchronizations in arrangements of interacting chaotic oscillators, and relate these conditions to the distribution of coupling weights and asymmetries in complex networks. We also analyze the conditions required for a high statistical occurrence of the same synchronization patterns, regardless of the oscillators’ initial conditions. Our results show that these persistent synchronization patterns are statistically more frequent in complex weighted networks than in regular ones, explaining why certain functional network topologies are often retrieved from experimental data. Besides, our results suggest that considering both the different coexisting synchronizations and also their statistics may result in a richer understanding of the relations between functional and structural networks of oscillators.

I. INTRODUCTION

Certain dynamical systems, which display oscillatory behavior in isolation, may display a wide repertoire of dynamical evolutions due to the coupling with their neighbors when embedded in networks of similar complex items.1 The relationship between network dynamics and structure in this type of systems is therefore a fundamental question in network science. For instance, the interaction of rhythmic elements may entail an adjustment of their oscillatory dynamics to finally end up in a state of (dynamical) agreement or synchronization.2–4 When coupling is strong, the oscillators in a network usually synchronize in a particular collective oscillatory behavior. However, for more moderate coupling intensity, this relationship may also be inhomogeneous, i.e., certain oscillators may synchronize whereas others may not.5–10 The specific patterns of synchronization, thus, provide information about the underlying couplings between the dynamical elements forming the network. Hence, a better characterization of the system can be achieved by analyzing all the synchronization relationships within a network instead of analyzing a single synchronization relationship. This type of characterization might be of crucial importance when the details of the contacts between the oscillators are not available.

In the past, studies of the synchronization patterns in networks of oscillators were mainly aimed at describing the conditions associated with the emergence of specific synchronization patterns in all the nodes.11 In the particular case of complex networks of coupled nonlinear oscillators, recent studies have provided evidence that it is possible to identify an appropriate interaction regime that allows to collect measured data to infer the underlying network structure based on time-series statistical similarity analysis,12 or connectivity stability analysis.13 In real-life systems, such as ecological networks,14 brain oscillations,15–18 or climate interactions,19 various types of complex synchronized dynamics have been observed to coexist in a single network. Therefore, such a diversity in dynamical relationships between the nodes

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endows a network with stability, flexibility, and robustness against perturbations.\textsuperscript{20}

The present work reveals that several types of stable synchronization patterns may coexist depending on the topology and on the distribution of coupling strengths within a network. Besides, the capacity of a network to display the same heterogeneous synchronization pattern regardless of the initial conditions, or consistency, is also investigated. Such a property, observed in some type of networks, allows to retrieve network structure from its dynamics in a more reliable way than using single synchronization patterns.

II. COEXISTENCE OF SYNCHRONIZATIONS

Consider two $n$-dimensional dynamical systems, $x$ and $y$, whose temporal evolutions are generally defined by $x(t) = F(x(t))$, $y(t) = G(y(t))$ in isolation. Assuming a bidirectional coupling scheme, the coupled system reads:

$$
\begin{align*}
x(t) &= F(x(t)) + \hat{C}(y(t) - x(t)), \\
y(t) &= G(y(t)) + \hat{C}(x(t) - y(t)).
\end{align*}
$$

(1)

Here, $x(t)$ and $y(t)$ are the $n$-dimensional state vectors of the systems, $F$ and $G$ are their corresponding vector fields, and $\hat{C}$ is a $n \times n$ matrix that provides the coupling characteristics between the sub-systems. When coupling is strong enough and these dynamical systems are oscillators, the synchronization relationships that can be established between them can be categorized in four types (see time traces in Fig. 1).\textsuperscript{22,23}

- Phase synchronization (PS) appears if the functional relationship between the dynamics of two oscillators preserves a bounded phase difference,\textsuperscript{24} with their amplitudes being largely uncorrelated. This can be exemplified by the relationship $|n\phi_1 - m\phi_2| < \text{const}$, with $n$ and $m$ being integer numbers which define the ratio between the phases $\phi_{1,2}$ of the two coupled oscillators.
- Generalized synchronization (GS) is observed if a complex functional relationship is established between the oscillators,\textsuperscript{25} e.g., $y(t) = H[x(t)]$, where $H[x(t)]$ can take any form other than identity. It can be thought to be a generalization of complete synchronization (CS) for non-identical oscillators.
- Lag synchronization (LS) appears when the amplitude correlation is high while at the same time there is a time shift in the dynamics of the oscillators,\textsuperscript{26} $y(t) = x(t - \tau)$, with $\tau$ being a lag time.
- Complete synchronization (CS) is observed when the coupled oscillators are identical or almost identical,\textsuperscript{27} and $x(t) = y(t)$ for a sufficiently large coupling strength $\hat{C}$.

There are several analysis techniques that can be used to assess the emergence of each of the mentioned synchronization motifs. Here, three of them are combined: cross-correlation (CC), Phase-Locking Value (PLV), and the Nearest-Neighbor Method (NNM). CC computes the lagged similarity or sliding

![FIG. 1. Heterogeneous synchronization patterns in complex weighted networks. (a) Examples of synchronization patterns (no synchronization NS, phase synchronization PS, generalized synchronization GS, lag synchronization LS, and complete synchronization CS) displayed by bidirectionally coupled Rössler oscillators. The upper panels show examples of $x(t)$ time traces for each synchronization pattern and the lower panels show examples of the corresponding delay-embedding plot.\textsuperscript{21} $\tau$ is the delay time for maximal cross-correlation in LS and PS. Examples of (b) a scale-free (SF) network ($K = 0.4$), (c) small-world (SW) network with low rewiring probability ($K = 0.1$), and (D) random (RN) ($K = 0.1$) of coupled Rössler oscillators displaying heterogeneous synchronization patterns. All networks have $N = 100$ nodes, $K$ is a global coupling parameter controlling the maximum coupling strength between two adjacent nodes (see Eq. (3)). For each type of network the right panels show the distribution of the coupling strengths $\xi_{ij}$ between pairs of nodes (upper panel) and the distribution of the synchronization patterns (polar histogram, lower panel). Each link is color-coded so as to show which synchronization pattern is displayed by each pair of oscillators within the network (NS, PS, GS, LS, CS see left bottom legend).]
The dynamics of each node $i$ follow the 3-dimensional Rössler equations, which read

$$
\begin{align*}
\dot{x}_i &= -\omega_i y_i - z_i + K \sum_{j=1, j \neq i}^{N_{\text{neigh}}} x_{ij} (x_j - x_i), \\
\dot{y}_i &= \omega_i z_i + ay_i, \\
\dot{z}_i &= p + z_i (x_i - c),
\end{align*}
$$

(3)

where $K$ is a global parameter controlling the maximum coupling strength between two nodes and $\omega_i$ is the natural frequency of the node $i$, which is normally distributed with average $\langle \omega \rangle = 1$ and standard deviation $\sigma_0 = 0.02$. An isolated node with Rössler dynamics can display periodic, quasi-periodic, or chaotic dynamics, and we choose $a = 0.15$, $p = 0.2$, and $c = 10$ to set the oscillators into a chaotic regime. \footnote{The coupling weights are set to depend on the number of neighbors of each node, if not specified otherwise, as

$$
\alpha_{ij} = \frac{1}{\sqrt{\text{deg}(v_i)\text{deg}(v_j)}},
$$

(4)

for $i \neq j$, where \text{deg}(v_i), \text{deg}(v_j)$ are the degrees (number of coupled neighbors) of two coupled nodes $v_i, v_j$.}

Figures 1(b)–1(d) show the distribution of synchronizations in three prototypical networks (composed of $N = 100$ nodes), namely, SF, SW, and RN, alongside with their weight distributions (relative frequency of $x_j$) and the distribution of synchronizations within each network. All three networks are located in a region of the coupling parameter space which allows a complex distribution of synchronizations, in between a non-synchronized and an all-synchronized network scenario. We call this type of behavior \textit{coexistence} of synchronization patterns. In this sense, the SF network shows clusters of PS, LS, and CS, and SW and RN networks show clusters of PS, GS, and LS, allowing for functional relationships between the oscillators. However, such distribution is very sensitive to the coupling characteristics and the underlying topology. Here, we show results for small and medium size networks. Naturally, the question of how our results scale with the network size is raised. The results shown here and other not shown indicate that what is relevant for the presence or absence of coexistence of synchronizations in a network is the distribution of couplings and not so much the size of the network. Further studies will have to explore, in detail, the dependence of coexistence (and consistency) of synchronization patterns with the network size. A proper characterization of the phenomenon requires the detailed analysis of the interaction of the oscillators’ dynamics and the networks they are embedded in.

### III. CONSISTENCY OF SYNCHRONIZATIONS

The heterogeneous synchronization motifs that emerge in complex networks are an excellent probe to detect the functional connectivity between the oscillators in a network. Besides, if these motifs are dynamically stable, synchronized states that show up recurrently when changing initial conditions might be identified, thus becoming an \textit{invariant} feature of the dynamics of the network. We are going to show that the attractor’s basin for specific coexistent synchronization patterns will depend on the topology of the network. So, in this section, we explore the conditions for the \textit{consistency} of synchronization patterns which we define as the persistence of the coexistent synchronization patterns regardless of the initial conditions.

A first example of coexistence of synchronizations is studied in a very simple weighted network formed by two pairs of nodes connected bidirectionally with a fifth node (see Fig. 2(a), Eq. (3)). The oscillators only differ on the frequencies, $\omega_i$, which are the following: $\omega_1 = 0.930$, $\omega_2 = 0.967$, $\omega_3 = 0.990$, $\omega_4 = 0.950$, and $\omega_5 = 0.970$. After fixing $x_{12}$ and $x_{34}$, the synchronization coexistence within the network can be changed by increasing the bidirectional coupling $x_c$ with the central node. Notice that the synchronization states evolve without changing $x_{1,2}$ and $x_{3,4}$ (the peripheral nodes’ coupling strengths).

Since non-identical oscillators are taken into account, there is no global synchronization manifold and, therefore, an \textit{analytical} stability analysis of the whole system cannot be performed. However, the evolution of the coexistence of synchronized states in terms of $x_c$ may be tracked numerically.
by considering in detail the values of the Conditional Lyapunov Exponents (CLEs, $\lambda 1_{\alpha i}$), indicating the onset of a different synchronization motifs. Lyapunov exponents are a measure that characterizes the stability or instability of the evolution of a dynamical system with respect to varying initial conditions or perturbations. For two unidirectionally coupled oscillators, $x(t)$ and $u(t)$ of dimensions $N_c$ and $N_p$, respectively, in which $x(t)$ drives $u(t)$, one can consider the presence of a time-dependent functional relationship

$$u(t) = H[x(t)].$$  \hspace{1cm} (5)$$

The dynamics of this coupled drive-response system is characterized by the Lyapunov exponent spectra $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_c}$ and $\lambda_1^u \geq \lambda_2^u \geq \cdots \geq \lambda_{N_p}^u$, with the latter being conditional Lyapunov exponents. In this sense, the rate of convergence or divergence of the trajectory of oscillator $u$ towards the trajectory defined by oscillator $x$ is given by $\lambda_k^u$; if $\lambda_k^u > 0$ the trajectories diverge, whereas if $\lambda_k^u < 0$ they converge.

Since throughout this manuscript a mutual coupling scheme is considered, Eq. (5) no longer holds for all time $t$, but rather its implicit form $H[x(t)] \cdot u(t) = 0$. However, locally (i.e., for $t^* - \delta < t < t^* + \delta$, with $\delta$ being infinitively small), the implicit function theorem allows to write $x(t^*) = H[u(t^*)]$ for other moments in time $t$. Therefore, without loss of generality, the spectrum of Lyapunov exponents can be computed in terms of the trajectory defined by one of the mutually coupled oscillators, either $u$ or $x$, as in the unidirectional coupling case. In what follows, the evolution of the flow of the trajectories of the coupled Rössler oscillators with respect to the trajectory defined by one of the oscillators in the networks is considered for small networks. This calculation allows to estimate whether such trajectory is attractive (i.e., neighboring oscillators converge to it and therefore synchronize) or repulsive (i.e., neighboring oscillators diverge from it and desynchronize in amplitude).

Figure 2(b) shows that, in terms of $x_c$, three different regions may be defined for the 5 (realization-averaged) largest CLEs, $\lambda 1_{\alpha i}$:

- In the first region ($0 < x_c \leq 0.06$) all the largest CLEs, are positive. The pairs 1–2 and 3–4 are mostly in PS. When increasing $x_c$ in this region, peripheral nodes become PS with the central node until the first 0 crossing of $\lambda 1_{\alpha 0}$ (light red line), which defines the onset for LS for pair 1–2 (vertical dashed line, first arrow, $x_c = 0.07$).
- The second region ($0.07 \leq x_c < 0.23$, in between dashed lines) sets a cascade of coexistence of synchronization regimes, i.e., successive zero-crossings of CLEs, determine the onset of GS and LS between the nodes. Notice that the heterogenous pattern PS/GS is rare and pattern GS/LS was never observed.
- In the third region, after $x_c = 0.23$, there is the onset of LS for the whole network.
Figure 2(c) shows the histogram of the occurrence of each pair of synchronized states between nodes 1–2 or 3–4, computed using CC, PLV, and NNM: in the coexisting region, there exist extended $\varepsilon_c$ values for which pairs 1–2 and 3–4 are, simultaneously, in several different synchronized regimes, e.g., 1–2 are in LS meanwhile nodes 3–4 are in PS. Therefore, for this range of coupling $\varepsilon_c$, several synchronized states can coexist in the network.

The cascade of zero-crossings of the CLEs, in terms of $\varepsilon_c$, can be expanded or squeezed by increasing or decreasing the symmetries of the system, and therefore the range of $\varepsilon_c$ values for which coexistence appears. For a completely symmetrical system, i.e., equal governing equations for all the nodes in a symmetric network, there are abrupt transitions to synchrony, without coexistence. Symmetry can be broken in a controlled way by means of a parameter governing the dynamics (e.g., oscillatory frequency), a parameter responsible for the topological characteristics of the network (e.g., clustering), or both features. In such scenarios different motifs of synchronized dynamics may show up, but they are restricted to a tiny region of the parameter space and, hence, appear to be spurious. Here, symmetry is broken by adding mismatches between the frequencies of the oscillators and by increasing the heterogeneity of the nodes’ degrees as well as the coupling values $\varepsilon_{ij}$.

Figure 3(a) shows the motif studied previously, but with different coupling strengths between peripheral nodes; $\varepsilon_{1,2}$ is now one order of magnitude smaller than $\varepsilon_{3,4}$ (see caption of Fig. 3), making this motif more asymmetrical in terms of coupling strength. Again, the evolution of the CLEs, is tracked for increasing $\varepsilon_c$ values. First, for $\varepsilon_c = 0$, nodes 1–2 are in PS meanwhile nodes 3–4 are in GS—i.e., a coexistence situation. As can be seen in Fig. 3(a), for different initial conditions zero-crossings of CLEs, appear along an extended $\varepsilon_c$ value region. In this case, the coexistence region for peripheral nodes 1–2 and 3–4 spans from $\varepsilon_c = 0$ to $\varepsilon_c = 0.20$. The right panel in Fig. 3(a) shows a plot of the relative frequency of synchronizations found for each pair of nodes in the small motif for $\varepsilon_c = 0.04$ (three pointed star) and $\varepsilon_c = 0.18$ (dotted circle). In the first case, $\varepsilon_c = 0.04$, each pair in the network lays in the same synchronization state for any of the imposed initial conditions, whereas in the second case, $\varepsilon_c = 0.18$, many pairs display different synchronizations depending on the initial conditions. Consequently, the first case displays more consistency than the second case because the network shows the same coexistence pattern regardless of the initial conditions.

Figure 3(b) shows a more symmetric network, in terms of coupling strength $\varepsilon_c$. Such relay configuration is less prone to synchronize for small coupling strengths and, therefore, larger $\varepsilon_c$ values are required to set synchronized states (see inset $\varepsilon_c = 0.18$). Fig. 3(b) lower right panel shows the relative frequency of synchronizations, with no large predominance of a single synchronization motif for a given pair of nodes. Therefore, the motif can be considered non-consistent.

Figure 3(c) shows an all-to-all small network in which all edges are weighted by the control parameter $\varepsilon_c$. In this case the network topology and the coupling strength distribution make this network more symmetrical. Accordingly, the $\varepsilon_c$ range for which coexistence exists is narrower with respect to the previous studied motifs. This reduction of the area of coexistence has implications in the consistency of synchronizations: zero-crossings of CLEs, are randomly distributed in a tiny range of $\varepsilon_c$ and, so, coupled pairs in the network do not consistently lay in the same synchronized state for different initial conditions (see Fig. 3(c) right panel).

Overall, by gathering the results of the coexistence and the consistency phenomena, we show that network symmetries govern the synchronization dynamics emerging from a system of coupled dynamical units. In this regard, clusters of synchronizations dynamically emerge thanks to symmetry breaking (with respect to the topology, the system parameter values, or both) and the statistics of the synchronization dynamics strongly depend on the type of symmetry breaking.

IV. CONSTRUCTION OF CONSISTENT NETWORKS

Functional networks can be constructed by establishing relationships between their (coupled) elements. One of the most prominent dynamical features that functionally relate two oscillators is synchronization, which may take the aforementioned forms (PS, GS, LS, and CS) among others not studied here. Therefore, synchronization is a probe for assessing a (non) trivial relationship between two dynamical systems. In this sense, in contrast to traditional approaches where only one type of synchronization is considered, the statistics of coexistence may reveal a complex functional organization of synchronization within a network and, therefore, may help to construct robust functional networks.

First, the motifs studied in Fig. 3 are coupled through their hubs (or most connected nodes) to construct a larger network of dynamical units. The resulting graph is shown in Fig. 4(a), where each of the motifs is labeled as A, B, or C. The intra-motif weights are the same as the selected in Figs. 3(a)–3(c), respectively, whereas the inter-hub links weights are shown in the caption of Fig. 4. Figure 4(b) shows the statistics of synchronization occurrence in this network: cluster A shows a very robust consistency of its synchronizations whereas clusters B and C are much less consistent, i.e., they display a wide repertoire of different synchronization motifs depending on the initial conditions. However, as can be noticed when comparing the relative-frequency plots shown in Figs. 3(a)–3(c) and 4(b), the dynamics of synchronization is altered when the three motifs are embedded in a larger network. This fact is a signature for assessing that the dynamics of coexistence in the large network is not just the simple juxtaposition of the dynamics of its composite sub-network motifs.

The construction of the functional networks arising from the synchronization patterns in this network is performed as follows: the statistical occurrence of each synchronization among pairs of nodes of the system is taken into account to better characterize the most salient synchronization motifs between the nodes. Then, thresholds in the statistical occurrence of each pairwise synchronization are applied, leading to the extraction of the links which, statistically, appear the most and so are more consistent.
Figure 4(c) shows the construction of the functional network emerging from the structural motif-based network by applying different levels of consistency for each synchronization pattern. For each threshold, this construction takes into account links that show the same synchronization a number of times greater than the consistency threshold. Accordingly, the constructed functional network coincides with the most consistent motif (A) rather than with subsystems B and C, which do not show consistent synchronization patterns. However, these non-consistent patterns allow us to infer a homogeneous coupling distribution in these modules, which is also informative about the underlying network structure.

The study of larger networks allows to generalize the relationship between the dynamical features of heterogeneous synchronization patterns with their structural and functional topological characteristics. By taking the SF prototypical network shown in Fig. 1 and performing topological changes—taking clustering as a control parameter—a
potential relationship between network symmetries and the consistency of the synchronization patterns can be unveiled.

Figure 5(a) shows the fraction of connected synchronized pairs in the SF networks whose consistency is above a certain threshold for increasing clustering. Noticeably, only low clustering networks have edges whose synchronization is consistent above a 50% of the realizations, arguably because only low clustering SF networks are heterogeneous enough to hold consistent synchronized dynamics. Figure 5(b) shows an example of a very consistent realization-averaged SF network with clustering $C = 0.15$ and a consistency map displaying the statistics of synchronization for each pair of nodes in the network. According to the statistics, the realization-averaged colors in the network mostly coincide with pure synchronization colors. Figure 5(c) shows a low consistency realization-averaged SF with clustering $C = 0.40$. The consistency map, performed for every pair of nodes in this network, shows no pattern compared with the case in panel (b). Such patterns denote that the functional organization of these networks is robust in the first case, whereas for the network with larger clustering randomized functional relationships are established among pairs of (connected) nodes.

The structure-function relationship can be quantified by calculating the number of coincident structural and functional links (called $n_{\text{true}}$ here onwards) and the number of non-coincident structural and functional links (called $n_{\text{false}}$ here onwards)

$$n_{\text{true}} = \frac{n_e}{n_T},$$

$$n_{\text{false}} = \frac{n_{\text{in}}}{n_{a-t-a} - n_T},$$

where $n_T$ is the number of edges in the structural network, $n_{a-t-a}$ is the number of edges in an equivalent all-to-all network, $n_e$ is the number of constructed edges that belong to the structural network, $n_{\text{in}}$ is the number of constructed edges that do not belong to the structural network, $n_{\text{true}}$ is the ratio of constructed edges that belong to the structural network, and $n_{\text{false}}$ is the ratio of constructed edges that do not belong to the structural network. In other words, $n_{\text{true}}$ computes how many of the structural edges have been reconstructed, whereas $n_{\text{false}}$ computes how many of the non-structural edges have been reconstructed. Note that the sum $n_{\text{true}} + n_{\text{false}}$ is not equal to 1 necessarily. In this sense, a construction with high $n_{\text{true}}$ and high $n_{\text{false}}$ indicates that the constructed network is close to an all-to-all network, i.e., all structural edges can be retrieved but the number of non structural edges is also high, implying a bad matching between structure and function. Figure 5(b) indicates that for clusterings below $C = 0.15$ the matching between structural and functional network is high for a consistency threshold of about 50%, whereas the construction for higher clusterings provides either a high ratio of false positives (close to all-to-all functional network) or non-consistent networks. Interestingly, the system faces a transition point at a relatively low clustering value, $C \approx 0.21$, which prevents the construction of functional networks at higher clusterings. Indeed, as the heterogeneity in the structural network is progressively lost due to higher clustering, the system loses consistency in the synchronization motifs and so no robust functional relationships can be extracted.

V. CONCLUSIONS

The coexistence of synchronizations is a phenomenon in which a variety of complex functional relationships are established between the dynamical evolutions of some coupled oscillatory elements. Such scenario emerges in the route towards an all-synchronized network, where trivial
correlations are established among oscillators. Although many patterns of synchronization have been described so far, like chimera states or single synchronization clusters, no description of the emergence of diverse amplitude and phase correlations within the same network has been addressed before. Therefore, the conditions for which this phenomenon occurs are given here in terms of the CLEs, providing a notion of the attractiveness of the trajectory defined by one oscillator. The coexistence of synchronizations is prominent when there is a broad separation of CLEs between positive and negative values, taking coupling strength as a control parameter. Besides, breaking the symmetry of a network—the number of nodes’ contacts or the coupling strengths—increases the even distribution of CLEs and allows for a broad distribution of synchronization motifs. Therefore, weighted complex networks show much more coexistence than their unweighted counterparts.

What is more, some networks can robustly display the same coexistence patterns regardless of the initial conditions imposed, showing high consistency. Such feature allows to better characterize the stable functional relationships established in the network, which is at the basis of functional network construction. In this sense, we show that the matching between structural and functional networks is high in networks displaying consistent heterogeneous synchronization states.

The consistency of the three prototypical networks shown in Fig. 1 is diverse: while SF networks with low clustering show high consistency, SW and RN networks do not display this feature because in SW or RN networks the number of node contacts fluctuates less. The consistency of the coexistence is a consequence of the heterogeneity of the network: the dynamical heterogeneous synchronization clusters consistently lay in the same heterogeneous synchronization manifolds for any of the initial conditions imposed because the synchronized trajectories are always dominated by the most connected neighbors. Previous research shows that, in unweighted and undirected networks, for certain coupling regimes, there is an optimal matching between structural and functional networks. Here, these results are extended to the case of weighted undirected networks and provide a novel method to extract robust functional relationships. Arguably, the method presented in this work is more restrictive because it relies on the preservation of the same heterogeneous
synchronization patterns, but it provides higher robustness to the constructed functional networks.

The present results, though limited in their scope, also point towards a general feature in the structure-function relationship in network science: the construction of functional networks, for the oscillators used here, bring about heterogeneous (non-symmetrical) networks because they are more consistent. More symmetric or homogeneous networks will appear as inconsistent if coupling is small; only when coupling is large enough to force global synchronization robust symmetrical networks will show up in the constructed functional networks. We believe that this structure-function relationship may also be true for other oscillators. Our result explains some previous experimental results. For instance, in brain dynamics, previous experimental studies have shown that consistent dynamics result in selected network topologies that have been retrieved much more often than others.\textsuperscript{36,37} However, further theoretical and experimental studies, for other systems, should address this point to limit or extend the validity of the conclusions raised here.

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