A total order in $[0,1]$ defined through a 'next' operator.

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1 Introduction.

Pierce expansions, see [2, 6, 7, 8, 9, 12, 13], permit the expression of any real number in $(0,1]$ through a series of the form:

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{k+1}}{a_1 a_2 \cdots a_k} + \cdots, \tag{1}$$

where $\{a_i\}$ is a strictly increasing sequence of positive integers. If the number of terms in (1) is finite, then α is a rational number; otherwise, the series represents an irrational number. In the finite case, to ensure the uniqueness of the representation, we require that the last two coefficients be not consecutive.

The expansion (1) will be denoted by:

$$\alpha = \langle a_1, a_2, \ldots, a_k, \ldots \rangle, \tag{2}$$

and we shall refer to the a_i as the *coefficients* or the *partial quotients* of the development.

As this representation model identifies a rational number in $(0,1]$ with a finite, strictly increasing sequence of positive integers, we used this fact in [6] to exhibit an actually computable enumeration of all positive rationals.

In this paper, we will use the mentioned ordering of the rationals in $(0,1]$ to define a *partial order* in the set $\mathbb{R}_1 = (0,1] - \{1 - 1/e\}$. This will be done through a 'next' operator:

$$\sigma : \mathbb{R}_1 \longrightarrow \mathbb{R}_1$$

which assigns a well–defined successor and predecessor for any number in $\mathbb{R}_1$. Then, with the help of the axiom of choice, the partial ordering will be transformed into a total ordering.

We will prove that, for any $\alpha \in \mathbb{R}_1$, the sequence $\{\sigma^n(\alpha)\}_{n\in\mathbb{Z}}$ is dense in $(0,1]$, and all its elements are of the same arithmetical character than α.

Lastly, the asymptotic distribution functions of the sequences formed by the half–orbits, $\{\sigma^n(\alpha)\}_{n\in\mathbb{N}}$, will be studied, proving all to be identical: a continuous, strictly increasing, singular function very similar to Minkowski's $?(\cdot)$ function, see [4, 11]. This is not a novelty, many interesting examples of singular functions come from considering the distribution of certain sequences, for instance, Erdös studied in this connection the sequence of fractional parts of $\log \Phi(n)/n$, see [1] and [3]. A good exposition on singular functions can be found in the excellent classic of Riesz & Nagy [10].

2 The enumeration of the rationals in $(0,1]$.

Let us reproduce briefly the enumeration described in [6]. We define

$$f : \mathbb{N} \longrightarrow Q^+$$

in the following way: if n is a positive integer whose dyadic expression is

$$n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_{k-1}} + 2^{a_k}, \qquad \text{with} \quad 0 \le a_1 < a_2 < \cdots < a_{k-1} < a_k,$$

then, in the case, $a_k - a_{k-1} > 1$ we define:

$$f(n) = \langle a_1 + 1, a_2 + 1, \dots, a_{k-1} + 1, a_k + 1 \rangle;$$

otherwise:

$$f(n) = \frac{1}{\langle a_1 + 1, a_2 + 1, \dots, a_{k-2} + 1, a_k + 1 \rangle}.$$

It is easy to see that f is a bijection that provides an enumeration for the rationals.

We restrict now this enumeration exclusively to the rationals in $(0, 1]$ which amounts to saying that we need to establish an ordering within the set of all finite, strictly increasing sequences of positive integers in which the last two terms are not consecutive. We will denote this last set by $\mathcal{P}_F^*(I\!N)$, and we will call its elements *admissible* sequences

Definition 1 *Given an admissible sequence,*

$$\{a_1, \dots, a_{k-1}, a_k\}, \quad \textit{with} \quad a_k - a_{k-1} > 1,$$

its successor will be the sequence defined as follows:

- *If* $a_1 > 1$, *then*
$$\sigma(\{a_1, \dots, a_k\}) = \{1, a_1, \dots, a_k\}.$$

- *If* $a_1 = 1$, *and* r *is the greatest integer such that* $a_r = r$, *then:*
$$\sigma(\{1, \dots, r, a_{r+1}, \dots, a_k\}) = \{r + 1, a_{r+1}, \dots, a_k\}.$$

- *In the case* $r = k - 1$ *and* $a_k = k + 1$, *then:*
$$\sigma(\{1, 2, \dots, k - 1, k + 1\}) = \{k + 2\}.$$

It is easy to see that the operator just defined, σ, generates from the sequence $\{1\}$, all other admissible sequences in $\mathcal{P}_F^*(I\!N)$, and consequently provides an enumeration of the rationals in $(0, 1]$, identical to the one described at the beginning of the section restricted to the unit interval.

3 Extension of the ordering to all real numbers in $I\!R_1$.

Let $\mathcal{P}^*(I\!N)$ be set formed by:

a) all strictly increasing sequences of positive integers, with the sole exception of the sequence formed by *all* positive integers;

b) all finite admissible sequences.

We are going to extend σ to $\mathcal{P}^*(I\!N)$.

Definition 2 *In the finite admissible case, σ operates on a sequence as we saw in definition 1. Now, given $\{a_1, a_2, \ldots, a_k, \ldots\}$, if $a_1 > 1$, then:*

$$\sigma(\{a_1, a_2, \ldots, a_k, \ldots\}) = \{1, a_1, a_2, \ldots, a_k, \ldots\};$$

otherwise, if r is the greatest integer such that $a_r = r$, then:

$$\sigma(\{1, \ldots, r, a_{r+1}, \ldots\}) = \{r+1, a_{r+1}, \ldots\}.$$

The exclusion of the sequence $a_n = n$ permits to ensure the existence of the integer r in the above definition.

The 'next' operator σ just defined in $\mathcal{P}^*(I\!N)$ can be immediately transferred to all reals in $I\!R_1$ just identifying strictly increasing sequences of positive integers with the real number whose Pierce expansion correspond to the sequence:

$$\begin{aligned} \sigma(\langle a_1, \ldots, a_k, \ldots\rangle) &= \langle b_1, \ldots, b_k, \ldots\rangle \iff \\ \sigma(\{a_1, \ldots, a_k, \ldots\}) &= \{b_1, \ldots, b_k, \ldots\} \quad \text{in} \quad \mathcal{P}^*(I\!N). \end{aligned} \tag{3}$$

The sole real number that would not have a successor would be

$$\langle 1, 2, 3, \ldots, n, \ldots\rangle = 1 - \frac{1}{e}.$$

4 An analytical expression for σ.

In the Pierce expansion of the number $1 - 1/e$:

$$1 - \frac{1}{e} = \langle 1, 2, 3, \ldots, n, \ldots\rangle$$

let us consider its *approximants* (truncations):

$$R_i = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{i+1}}{i!}.$$

We have the following infinite chain of inequalities:

$$0 = R_0 < R_2 < R_4 < \cdots < 1 - \frac{1}{e} < \cdots < R_5 < R_3 < R_1 = 1.$$

We can now consider the following family of half–open intervals, mutually disjoint, taken at left and right of $1 - 1/e$: on the left, $[R_{2k}, R_{2k+2})$ and on the right, $(R_{2k+1}, R_{2k-1}]$, such that, being mutually disjoint we have:

$$\begin{aligned} \bigcup_{k=0}^{\infty} [R_{2k}, R_{2k+2}) &= [0, 1 - \frac{1}{e}) \\ \bigcup_{k=1}^{\infty} (R_{2k+1}, R_{2k-1}] &= (1 - \frac{1}{e}, 1]. \end{aligned}$$

Theorem 4.1 *If R_i is the sequence of approximants of $1-1/e$, the function σ defined on $\mathbb{R}_1$, has the following analytical expression:*

$$\sigma(x) = \begin{cases} -\frac{(2k)!}{2k+1}(x - R_{2k}) + \frac{1}{2k+1} & \text{if } x \in [R_{2k}, R_{2k+2}) \quad (k = 0, 1, 2, \ldots); \\ \frac{(2k-1)!}{2k}(x - R_{2k-1}) + \frac{1}{2k} & \text{if } x \in (R_{2k+1}, R_{2k-1}] \quad (k = 1, 2, \ldots). \end{cases}$$

Proof.

We are going to find the expression of σ only for the intervals on the left of $1-1/e$, that is to say for the intervals of the form $[R_{2k}, R_{2k+2})$, as the procedure is the same for the intervals on the other side.

The Pierce expansion of the reals in (R_{2k}, R_{2k+2}) are:

$$\langle 1, 2, 3, \ldots, 2k, a_{2k+1}, \ldots \rangle \quad \text{where} \quad a_{2k+1} \geq 2k + 2.$$

The function σ operates on these numbers as we described in definition 2:

$$(4) \qquad \sigma(\langle 1, 2, 3, \ldots, 2k, a_{2k+1}, \ldots \rangle) = \langle 2k + 1, a_{2k+1}, \ldots \rangle.$$

This last equation can be written as:

$$(5) \qquad \sigma(R_{2k} + \frac{1}{(2k)!}\langle a_{2k+1}, \ldots \rangle) = \frac{1}{2k+1} - \frac{1}{2k+1}\langle a_{2k+1}, \ldots \rangle.$$

If we denote by x the expression $R_{2k} + \frac{1}{(2k)!} \cdot \langle a_{2k+1}, \ldots \rangle$, it is easy to see that with simple transformations we obtain:

$$\frac{1}{2k+1} - \frac{(2k)!}{2k+1}(x - R_{2k}) = \frac{1}{2k+1} - \frac{1}{2k+1}\langle a_{2k+1}, \ldots \rangle = \sigma(x).$$

We can sum up in the following way:

$$(6)\ x \in (R_{2k}, R_{2k+2}) \Rightarrow \sigma(x) = -\frac{(2k)!}{2k+1}(x - R_{2k}) + \frac{1}{2k+1} \quad (k = 0, 1, \ldots).$$

Thus σ is a linear function within the intervals considered. Let us see what happens in the end–points of the intervals.

Developing R_{2k} we have:

$$R_{2k} = \langle 1, 2, \ldots, 2k - 2, 2k \rangle,$$

and

$$\sigma(R_{2k}) = \frac{1}{2k+1},$$

according to the definition of σ. Now, this is exactly the value obtained with the expression (6) applied to R_{2k}; thus we can add this point to the domain of validity of (6):

$$x \in [R_{2k}, R_{2k+2}) \Rightarrow \sigma(x) = -\frac{(2k)!}{2k+1}(x - R_{2k}) + \frac{1}{2k+1} \quad (k = 0, 1, \ldots).$$

For the other end-point, R_{2k+2}, we find a jump discontinuity: on the one hand we have

$$\sigma(R_{2k+2}) = \frac{1}{2k+3}\,;$$

and, on the other:

$$\lim_{\substack{x\to R_{2k+2}\\ R_{2k}<x<R_{2k+2}}} \sigma(x) = -\frac{(2k)!}{2k+1}(R_{2k+2}-R_{2k}) + \frac{1}{2k+1} = \frac{1}{2k+2}.$$

4.1 The inverse function of σ.

It is seen at once that σ^{-1} exists and can be written as:

$$\sigma^{-1}(x) = \begin{cases} -\frac{1}{(2k)!}((2k+1)x-1) + R_{2k} & \text{if } x \in \left(\frac{1}{2k+2}, \frac{1}{2k+1}\right] \quad (k=0,1,\ldots) \\ \\ \frac{1}{(2k-1)!}(2kx-1) + R_{2k-1} & \text{if } x \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right] \quad (k=1,2,\ldots). \end{cases}$$

5 Ordering $I\!\!R_1$.

The orbits of σ, that is to say the sets of the form $\{\sigma^n(\alpha)\}_{n\in Z\!\!\!Z}$ form a partition of $I\!\!R_1$ and within each orbit, σ operates as a 'next' operator. We have thus a partial order immediately defined:

$$(7) \qquad \alpha \preceq \beta \quad \text{if it exists} \quad n \in I\!\!N,\ \sigma^n(\alpha) = \beta.$$

Such a partial order can be made total in $I\!\!R_1$ just by choosing a representative of each orbit (we need the axiom of choice to do that) which we will denote by f_α. The total order is now established as:

$$(8) \qquad \gamma \preceq \delta \Longleftrightarrow f_\gamma \le f_\delta,$$

when γ and δ do not belong to the same orbit.

Let us denote by $[f_\alpha]$ the equivalence class of f_α.

Theorem 5.1 *For all $\alpha \in I\!\!R_1$ we have:*

1. *If $\alpha \in Q$, then*
$$\forall \alpha_1 \in [f_\alpha] \Rightarrow \alpha_1 \in Q.$$
2. *If α is an algebraic irrational of degree k, then*
$$\alpha_1 \in [f_\alpha] \Rightarrow \alpha_1 \ \textit{is an algebraic irrational of degree } k.$$
3. *If α is transcendental, then*
$$\alpha_1 \in [f_\alpha] \Rightarrow \quad \alpha_1 \ \textit{is transcendental.}$$

The proof is immediate as both σ and σ^{-1} have linear expressions which preserve the arithmetical character of numbers.

It is worth mentioning in passing that the set $\{f_\gamma\}$ of all the orbit representatives is not Lebesgue measurable.

6 Density of the orbits.

Lemma 6.1 *If the Pierce expansions of two real numbers, α and β coincide from a determined position, they belong to the same orbit.*

Proof.
Let

$$\begin{aligned}\alpha &= \langle a_1, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots \rangle \\ \beta &= \langle b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots \rangle,\end{aligned}$$

with $a_{r+i} = b_{k+i}, \;\; i = 0, 1, \ldots$.
If $b_{k-1} > a_{r-1}$, then $\sigma^n(\alpha) = \beta$, where n:

$$n = 2^{b_{k-1}-1} + \cdots + 2^{b_1-1} - \left(2^{a_{r-1}-1} + \cdots + 2^{a_1-1}\right).$$

From the previous lemma, the following theorem is easily proved:

Theorem 6.2 *The closure of each orbit is* $[0, 1]$.

Proof.
In the case of the orbit of all rationals in $(0, 1]$, the result is obvious. Now, let $\alpha = \langle a_1, \ldots, a_n, \ldots \rangle$ be any irrational and let us consider its orbit $\{\sigma^n(\alpha)\}_{n\in\mathbb{Z}}$. Let γ be an irrational not belonging to the orbit of α, and let $\epsilon > 0$.
Let the Pierce expansion of γ be:

$$\gamma = \langle c_1, \ldots, c_k, \ldots \rangle.$$

As the sequence of the c_i is strictly increasing there has to exist a subscript r for which

$$\frac{1}{c_1 c_2 \cdots c_r} < \epsilon.$$

Let us consider now any element, α', of the orbit of α whose Pierce expansion starts with $c_1, \ldots, c_r$, which is always possible thanks to lemma 6.1. We have:

$$|\alpha' - \gamma| \leq \frac{1}{c_1 \cdots c_r} < \epsilon.$$

7 The asymptotic distribution function of an orbit.

In this section we are going to find the asymptotic distribution function (a.d.f.) of the sequence formed by the orbit of all rationals. We are going to prove that it is a function similar to Minkowski's $?(x)$ function, see [4]: a continuous singular function, strictly increasing transforming a set of measure zero into a set of measure one and viceversa. For the sake of completeness we remind the definition of an a.d.f. More details can be found in [5, page 53].

Definition 3 *A sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *of real numbers in* $[0,1]$ *is said to have the a.d.f.* $F(x)$ *if*

$$\lim_{n\to\infty}\frac{\#\{\alpha_i \leq x; \quad i=1,2,\ldots,n\}}{n} = F(x) \qquad \textit{for } 0\leq x\leq 1.$$

The two following lemmas are easy to prove:

Lemma 7.1 *If two sequences* $\{a_n\}$ *and* $\{b_n\}$ *belonging to* $[0,1]$*, coincide from a given position, then if one of them has an a.d.f. the other one has the same a.d.f.*

Lemma 7.2 *Given a sequence* $\{a_n\}$ *such that for a fixed* k*:*

$$\lim_{n\to\infty}\frac{\#\{a_i \leq x; \quad i=1,2,\ldots,kn\}}{kn} = F(x),$$

then $F(x)$ *is the a.d.f. of the sequence.*

Lemma 7.3 *Given the sequence* $q_n=\sigma^n(1)$*, the unitary fraction* $\langle m\rangle$ *occupies position* $n=1+2^{m-2}$*.*

Proof.

The number of rational numbers whose Pierce expansion ends with a given integer s is 2^{s-2}, where $s\geq 2$. Consequently all Pierce expansions ending with an integer less than m form a set of $\sum_{s=2}^{m-1}2^{s-2}+1=2^{m-2}$ elements. Then $q_{2^{m-2}+1}=\langle m\rangle$.

Lemma 7.4 *Given a positive integer* m *and a finite sequence of positive integers* $\{c_i\}$*, such that* $m<c_1<\ldots<c_{r-1}<c_r-1$ *we have that all the elements in the sequence* $\{q_i\}=\sigma^i(1)$ *of the form*

$$\langle b_1,\ldots,b_r,c_1,\ldots,c_r\rangle, \quad \textit{with} \quad b_r\leq m,$$

constitute a block of 2^m *consecutive elements.*

Proof.

The first element of the block is $\langle c_1,\ldots,c_r\rangle$, and the last is $\langle 1,\ 2,\ldots,m, c_1,\ldots,c_r\rangle$, so all in all we have as many elements as subsets of the set $\{1,2,\ldots, m\}$.

Lemma 7.5 *Given a positive integer* m*, the sequence* $\{q_i\}$ *can be organized in blocks of length* 2^m *in such a way that the first block is formed by all the rationals whose Pierce expansion ends with a integer less than* $m+2$*, and the rest of blocks has the composition indicated in lemma 7.4.*

Proof.

According to lemma 7.3 the element $\langle m+2\rangle$ is placed in position 2^m+1. Therefore the first 2^m elements of the sequence $\{q_i\}$ are formed by all the expansions ending with an integer less than $m+2$. The rest of blocks are formed by 2^m elements as they correspond to the requirements of lemma 7.4. Each block starts and ends with subscripts of the form $1+r2^m$ and $(r+1)2^m$.

Lemma 7.6 *If $x = \langle a_1 \rangle$, then*

$$\lim_{n\to\infty} \frac{\#\{\sigma^i(1) \le x; \quad i = 1, 2, \dots, n\}}{n} = \frac{1}{2^{a_1-1}}.$$

Proof.

Let m be an integer such that $m \ge a_1 + 2$. By lemma 7.3 we know that $q_{2^m+1} = \langle m+2 \rangle$. According to lemma 7.1 we know that the a.d.f. of $\{q_i\}$ coincides with that of $\{\bar{q}_i\}$, where $\bar{q}_i = q_{2^m+i}$, and whose first element is $\bar{q}_1 = \langle m+2 \rangle$.

By lemma 7.5 we can organize $\{\bar{q}_i\}$ in blocks of 2^m elements in such a way that the requirements of lemma 7.4 are fulfilled. For each of these blocks the terms of the sequence which are less or equal than $\langle a_1 \rangle$ will be of the form:

$$\langle b_1, \dots, b_r, c_1, \dots, c_s \rangle, \quad \text{with} \quad b_1 \ge a_1.$$

There will be as many of these as subsets of the set $\{a_1, \dots, m\}$, that is 2^{m+1-a_1}. Consequently, for each block we have:

$$\frac{\#\{\bar{q}_i \le x, \quad i = r2^m + 1, \dots, (r+1)2^m\}}{2^m} = \frac{2^{m+1-a_1}}{2^m} = \frac{1}{2^{a_1-1}}.$$

As the same is verified by all blocks we have:

$$\frac{\#\{\bar{q}_i \le x, \quad i = 1, \dots, k \cdot 2^m\}}{k \cdot 2^m} = \frac{1}{2^{a_1-1}}.$$

According to lemma 7.2 we finally have:

$$F(\langle a_1 \rangle) = \lim_{n\to\infty} \frac{\#\{\bar{q}_i \le x, \quad i = 1, 2, \dots, n\}}{n} = \frac{1}{2^{a_1-1}}.$$

Lemma 7.7 *If $x = \langle a_1, \dots, a_n \rangle$, then*

$$(9) \quad \lim_{n\to\infty} \frac{\#\{\sigma^i(1) \le x; \quad i = 1, 2, \dots, n\}}{n} = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_2-1}} + \dots + \frac{(-1)^{n+1}}{2^{a_n-1}}.$$

Proof.

We will use induction on the length of the Pierce expansion of x. Lemma 7.6 takes care of length 1. Let us suppose that (9) is true for an odd length, $n = 2r - 1$; we are going to prove that in this case, it is also true for $n = 2r$ and $n = 2r + 1$.

As we did to prove lemma 7.6, let $m \ge a_{2r+1}$, and let us organize our sequence in blocks of length 2^m. By the induction hypothesis, we will have that for $x_{2r-1} = \langle a_1, \dots, a_{2r-1} \rangle$ we have:

$$\frac{\#\{\bar{q}_i \le x, \quad i = 1 + (k-1)2^m, \dots, k \cdot 2^m\}}{2^m} = \frac{1}{2^{a_1-1}} - \dots + \frac{1}{2^{a_{2r-1}-1}}.$$

If we now consider $x_{2r} = \langle a_1, \dots, a_{2r} \rangle$, then the $\bar{q}_i \le x_{2r}$ in the block we are considering will be the same as before, $\bar{q}_i \le x_{2r-1}$, except for those of the form:

$$\langle a_1, \dots, a_{2r-1}, b_{2r}, \dots, b_{2r+i}, c_1, \dots, c_r \rangle,$$

where the b_j verify $a_{2r} \leq b_{2r} < \cdots < b_{2r+i} \leq m$. There are a total of $2^{m-a_{2r}+1}$ verifying this. So, in each block we would have:

$$\frac{\#\{q_i \leq x_{2r}\}}{2^m} = \frac{\#\{q_i \leq x_{2r-1}\}}{2^m} - \frac{2^{m-a_{2r}+1}}{2^m} = \frac{1}{2^{a_1-1}} - \cdots - \frac{1}{2^{a_{2r}-1}}.$$

In the same way, if we take $x_{2r+1} = \langle a_1, \ldots, a_{2r+1}\rangle$, the $\bar{q}_i \leq x_{2r+1}$ in a block are as many as those for x_{2r} plus those of the form

$$\langle a_1, \ldots, a_{2r}, b_{2r+1}, \ldots, b_{2r+i}, c_1, \ldots, c_k\rangle,$$

where the b_j verify $a_{2r+1} \leq b_{2r+1} < \cdots < b_{2r+i} \leq m$. Thus to the previously found quantity we have to add $2^{m-a_{2r+1}+1}$ obtaining a total of:

$$\frac{\#\{\bar{q}_i \leq x_{2r+1}\}}{2^m} = \frac{\#\{\bar{q}_i \leq x_{2r}\}}{2^m} + \frac{1}{2^{a_{2r+1}-1}}.$$

This ends the proof of lemma 7.7.

Theorem 7.8 *If $x = \langle a_1, a_2, \ldots, a_k, \ldots\rangle$ is irrational, then*

$$\lim_{n\to\infty} \frac{\#\{\sigma^i(1) \leq x;\;\; i = 1, 2, \ldots, n\}}{n} = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_2-1}} + \cdots + \frac{(-1)^{k+1}}{2^{a_k-1}} + \cdots.$$

Proof.

For x a rational number in $[0,1]$, let

$$F(x) = \lim_{n\to\infty} \frac{\#\{\sigma^i(1) \leq x; \quad i = 1, 2, \ldots, n\}}{n}$$

whose existence has been proved in lemmas 7.6 and 7.7. $F(x)$ is by its own nature a non–decreasing function. Thus if now we take x irrational, for any pair of its approximants we have,

$$R_{2k} < x < R_{2k+1} \Longrightarrow F(R_{2k}) \leq F(x) \leq F(R_{2k+1}).$$

And as we have

$$F(R_{2k+1}) - F(R_{2k}) = \frac{1}{2^{a_{2k+1}-1}}\,,$$

tending to 0 as $k \to \infty$, we have

$$F(x) = \lim_{n\to\infty} F(R_n) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_n-1}}.$$

Theorem 7.9 *All half–orbits, $\{\sigma^n(\alpha)\}_{n\in I\!N}$ have $F(x)$ as their a.d.f.*

Proof.

Let $\alpha = \langle a_1, \ldots, a_k, \ldots\rangle$ be any irrational. There exists a $n < 2^m$ such that $\sigma^n(\alpha) = \langle m+1, \ldots, a_i, \ldots\rangle$. Now, the same proof we have just used in the preceding lemmas can be applied to the sequence $\{\sigma^{n+i}(\alpha)\}$.

Exactly in the same way, the sequences of the form $\{\sigma^{-n}(\alpha)\}$ have also the same a.d.f.

8 The singularity of $F(x)$.

It is easy to see that $F(x)$ is continuous because any irrational $\beta \in [0,1]$ can be uniquely expanded in the form (alternated dyadic system):

$$\beta = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{b_n}}, \quad 0 \leq b_1 < \cdots < b_n < \cdots$$

(Rationals have finite expansions in the alternated dyadic system). This implies that $F(x)$ is one to one and onto, and being non–decreasing it has to be continuous. By Lebesgues's theorem it has a finite derivative almost everywhere. We have the following result:

Lemma 8.1 *On irrational numbers of the form $\langle a_1, \ldots, a_k, \ldots \rangle$, the derivative $F'(x)$, when it exists, takes the value:*

$$\lim_{k\to\infty} \frac{a_1 \cdot a_2 \cdots a_k}{2^{a_k - 1}}.$$

Proof.
Let $x = \langle a_1, \ldots, a_k, \ldots \rangle$ be an irrational. The sequence $\{R_i\}$ of its approximants verify

$$F(R_{2k}) < F(x) < F(R_{2k+1}).$$

The derivative, when it exists, coincides with the limit:

$$F'(x) = \lim_{k\to\infty} \frac{F(R_{2k+1}) - F(R_{2k})}{R_{2k+1} - R_{2k}} = \frac{\frac{1}{2^{a_{2k+1}-1}}}{\frac{1}{a_1 \cdots a_{2k+1}}} = \frac{a_1 \cdots a_{2k+1}}{2^{a_{2k+1}-1}}.$$

Theorem 8.2 $F'(x) = 0$ *almost everywhere.*

Proof.
Let K be the set of the $x \in (0,1]$, such that their Pierce expansion verify:

$$\lim_{k\to\infty} \frac{\log a_k(x)}{k} = 1. \tag{10}$$

As Shallit proved in [12], K is a set of measure 1. Let H denote the set where $F'(x)$ exists, whose measure is also 1. Let $x \in K \cap H$. Condition (10) implies that given $\epsilon > 0$ there exists a subscript n_0, $(n_0(x))$, such that for $n \geq n_0$ the following inequality is verified:

$$e^{n(1-\epsilon)} < a_n(x) < e^{n(1+\epsilon)},$$

Let us see that $F'(x) = 0$. We have

$$F'(x) = \lim_{k\to\infty} \frac{a_1(x) \cdots a_k(x)}{2^{a_k(x)-1}}.$$

Let us denote by

$$b_k(x) = \frac{a_1(x) \cdots a_k(x)}{2^{a_k(x)-1}}.$$

For $k \geq n_0$:

$$\begin{aligned}
b_k(x) &= \frac{2 \cdot a_1(x) \cdots a_k(x)}{2^{a_k(x)}}, \\
\log b_k(x) &= \log 2 + \log a_1(x) + \cdots + \log a_k(x) - a_k(x) \log 2 \leq \\
&\leq \log 2 + k \cdot \log a_k(x) - a_k(x) \log 2, \\
\log b_k(x) &\leq \log 2 + k^2 \cdot \frac{\log a_k(x)}{k} - e^{k(1-\epsilon)} \log 2 = \\
&= \log 2 + k^2 \underbrace{\left(\frac{\log a_k(x)}{k} - \frac{e^{k(1-\epsilon)} \log 2}{k^2} \right)}_{(\star)}.
\end{aligned}$$

As the first term of $(\star)$ tends to 1 when $k \to \infty$ and the second term tends to $-\infty$, we have:

$$\lim_{k \to \infty} \log b_k(x) = -\infty, \quad \Rightarrow \quad \lim_{k \to \infty} b_k = 0$$

and, consequently, $F'(x) = 0$.

In the previous proof, x can be considered in a much larger set than K. The same proof can be applied to any $x \in H$ whose Pierce expansion verify:

$$\lim_{n \to \infty} \frac{\log a_n(x)}{n} = r > 0.$$

If $x \in H$, a necessary (but not sufficient) condition for $F'(x) > 0$ is:

$$\lim_{n \to \infty} \frac{\log a_n(x)}{n} = 0.$$

9 $F(x)$ and the metrical properties of two systems for representing real numbers.

As we have seen, $F(x)$ relate two systems of representation: the one based on Pierce expansions and the alternated dyadic. This bridge between both systems permits an analysis of the metrical properties of both models. Let us state a result in that sense:

Theorem 9.1 *The a.d.f. $F(x)$ transforms a set of measure 1 into a null set.*

Proof.

Let K and H be defined as before. If $x \in K \cap H$, its image $F(x)$ is:

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_n(x)-1}}. \tag{11}$$

This last expression is the development of a number in the alternated dyadic system, where the $a_n(x) - 1$ can be considered its 'digits'. In this system, an irrational number has a unique expression of the form:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{d_n(x)-1}} \, ,$$

which can also be written as a list of 0's and 1's. For instance,

$$\frac{1}{2^2} - \frac{1}{2^5} + \frac{1}{2^6} - \cdots = [0, 0, 1, 0, 0, 1, 1, \ldots]$$

with 1 in place $d_n(x)$ and 0 elsewhere, signs alternating.

Going back to our images, $F(x)$, as found in (11), these images are not 'normal' in the alternated dyadic system, understanding by normal a number whose 'digits', $d_n(x)$, verify:

$$\lim_{n\to\infty} \frac{n}{d_n(x)} = \frac{1}{2}. \tag{12}$$

Equation (12) has to be interpreted as the distribution of 1's in the expression of a normal number: n is the number of 1's in the first $d_n(x)$ places.

Equivalently, we can write (12) as an asymptotic equality:

$$d_n(x) = 2n + o(n),$$

and hence:

$$\lim_{n\to\infty} \frac{\log d_n(x)}{n} = 0.$$

This shows that a necessary condition for $F(x)$ to be normal in the alternated dyadic system is:

$$\lim_{n\to\infty} \frac{\log a_n(x)}{n} = 0.$$

The condition is not sufficient.

This analysis shows that the image of $K \cap H$ by $F(x)$, is a subset of the set of not–normal numbers in the alternated dyadic system and, consequently,

$$\lambda(K \cap H) = 1, \quad \lambda(F(K \cap H)) = 0,$$

where λ is the usual Lebesguian measure.

In the same way we could prove that the set M of the $x \in (0, 1]$ such that their Pierce expansions verify:

$$\sum_{n=1}^{\infty} \frac{1}{a_n(x)} = \infty,$$

which is null set as Shallit proved in [12], is transformed by $F(x)$ into a set of measure 1 as in the alternated dyadic system the property just mentioned is antithetic to the corresponding property in the Pierce expansion model, obtaining

$$\lambda(M) = 0, \quad \lambda(F(M)) = 1.$$

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