Quotient Spaces of Boundedly Rational Types*

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Abstract

By identifying types whose low-order beliefs – up to level \(\ell_i\) – about the state of nature coincide, we obtain quotient type spaces that are typically smaller than the original ones, preserve basic topological properties, and allow standard equilibrium analysis even under bounded reasoning. Our Bayesian Nash \((\ell_i, \ell_{-i})\)-equilibria capture players’ inability to distinguish types belonging to the same equivalence class. The case with uncertainty about the vector of levels \((\ell_i, \ell_{-i})\) is also analyzed. Two examples illustrate the constructions.

Keywords: Incomplete-information games, high-order reasoning, type space, quotient space, hierarchies of beliefs, bounded rationality. JEL Classification: C72, D03, D83.

1 Introduction

Starting with Harsanyi’s (1967-1968) seminal work, players’ private information is conveniently represented by types, which correspond to infinite hierarchies of beliefs. In many applications, the predictions of standard game theory can be heavily dependent on the specification of high-order beliefs. As a matter of fact, Weinstein and Yildiz (2007a) show that any rationalizable action (Battigalli and Siniscalchi, 2003, and Dekel, Fudenberg, and Morris, 2007) of any type can be made into the uniquely rationalizable action for an open set of perturbed types under an appropriate class of beliefs perturbations. This generalizes the intuition provided by Carlsson and van Damme (1993) in their global games. Nonetheless, experiments have highlighted that high-order beliefs might have a less large relevance in practice (see the literature mentioned in the context of the examples given in Sections 3.1.1 and 4.1.1). Weinstein and Yildiz (2007b) write that bounds on rationality translate into bounds on beliefs, and consequently situations where high-order beliefs are decisive might necessitate a change of paradigm.

In this paper we provide an approach towards relaxing the (implicit) assumptions made on players’ ability to keep track of high-order beliefs. The approach we propose consists of defining certain quotient type spaces,

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the elements of which gather – or “amalgamate” in the usage of Aumann and Dreze (2008, Section VI) – types that have the same low-order beliefs, where low means up to some order $\ell_i$. Such quotient spaces will be referred to as $\ell_i$-quotient type spaces. The underlying behavioral hypothesis is that a player acts as if he could not tell his own types apart that have identical belief hierarchies from order 1 to $\ell_i$. Players hence condition their interim expected payoff on their type belonging to its equivalence class rather than on the type itself.

We would like to stress that our quotient type spaces embody one specific behavioral hypothesis. A researcher interested in modeling other aspects of human behavior can adopt different notions of similarity between types and possibly define quotient type spaces accordingly.

Standard equilibrium analysis can be performed by means of $\ell_i$-quotient type spaces: in an $(\ell_i, \ell_{-i})$-equilibrium each player $i$ plays a strategy that is constant within his equivalence classes and maximizes his expected payoff. If $(\ell_i, \ell_{-i})$ is uncertain, we assume a commonly known common prior $\lambda$ on levels and let players with different quotient spaces be different players that possess the same information as the original players from whom they are derived: in a $\lambda$-equilibrium each of these new players has a strategy that is constant within his equivalence classes and maximizes the expectation of his payoff with respect to the state of nature, the other players’ type, and the other players’ levels $\ell_{-i}$ as well.

The way in which we introduce uncertainty about players’ levels of sophistication deserves a few comments. In the class of games we consider, we have a common prior $f$ on the state of nature, player $i$’s type, and player $-i$’s type – respectively denoted by $\theta$, $t_i$, and $t_{-i}$. We can think of $f$ as the marginal of another distribution – denote it $F$ – on $\theta$, $t_i$, $t_{-i}$, $\ell_i$, and $\ell_{-i}$. Then, $\lambda$ is the marginal of $F$ on $\ell_i$ and $\ell_{-i}$. We assume that $F$ is given by the product of $f$ and $\lambda$, which is equivalent to $(\ell_i, \ell_{-i})$ being independent of $(\theta, t_i, t_{-i})$.1

In the original game, a type $t_i$ has a posterior on $\theta$ and $t_{-i}$, but once we allow for the possibility of boundedly rational players, a type $t_i$ of level $\ell_i$ should have a posterior on $\theta$, $t_{-i}$, and $\ell_{-i}$. Indeed, $t_i$ needs to know what $-i$ is conditioning his expected payoff on in order to outguess what action $-i$ is going to take. The question is how to formalize this idea. One could let player $i$’s strategy map each $(t_i, \ell_i)$ into an action. Otherwise, one could: (i) as usual, let player $i$’s strategy be a map that associates an action to each $t_i$; (ii) enlarge the set of players and treat a player $i$ of level $\ell_i$ and any player $i$ of level $\ell_i' \neq \ell_i$ as distinct players. Since our primary objective is to provide researchers with a familiar and parsimonious tool for dealing with bounded rationality in games with incomplete information, we opted for the latter solution. By means of suitable transformations of the payoff functions, we obtain that in our $\lambda$-equilibria players do not have to form beliefs about the levels of sophistication (i.e., player $i$ does not have to think about $\ell_{-i}$, about what $-i$ thinks about $\ell_{-i}$, and so forth). Importantly, this does not prevent us from deriving results that match behavioral patterns found in the laboratory.2

As an illustration of the theory, we consider two well-known examples where high-order beliefs play a key role, namely, Rubinstein’s (1989) electronic mail game3 and Morris and Shin’s (2002) game with private and public information. We obtain, respectively, $(\ell_i, \ell_{-i})$-equilibria and $\lambda$-equilibria that appear to be closer to experimental evidence than usual game-theoretical predictions.

The paper is organized as follows: Section 2 sets up the notation; Section 3 introduces the $\ell_i$-quotient space and computes $(\ell_i, \ell_{-i})$-equilibria for the electronic mail game; Section 4 presents the case where players’

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1In Section 4.1 we explain the technical reasons that motivate such an assumption.

2We are unaware of whether the two modeling choices are equivalent or one is more expressive than the other.

3See also Halpern (1986) on the coordinated-attack problem in computer science.
levels of sophistication \((\ell, \ell_{-i})\) in distinguishing types are uncertain and computes Morris and Shin’s (2002) game with private and public information; Section 5 concludes by summarizing related research.

## 2 Preliminaries

We consider two-player static games of incomplete information. Players are denoted by \(i, -i \in I = \{1, 2\}\), where \(-i = j \in I\) such that \(j \neq i\). The concepts developed straightforwardly extend to the case of finitely many players \(2 \leq |I| < \infty\). Our setup builds on Siniscalchi (2008). At the center of the construction is the set of all states of nature \(\Theta\) consisting of payoff-relevant information players are uncertain about. A typical element of \(\Theta\) is denoted by \(\theta\).

For an arbitrary space \(S\), let \(\Delta(S)\) denote the set of probability measures on the Borel \(\sigma\)-algebra of \(S\), endowed with the weak* topology. Product spaces are always endowed with the product topology and subspaces with the subspace topology. Countable spaces are endowed with the discrete topology. Given a Polish (i.e., separable and completely metrizable) space \(\Theta\), let \(X^0 = \Theta\) and for \(\ell \geq 1\) define recursively \(X^\ell = X^{\ell-1} \times \Delta(X^{\ell-1})\). Let \(H = \prod_{\ell \geq 1} \Delta(X^{\ell-1})\) be the space of all possible belief hierarchies with typical element \(\mu = (\mu^1, \mu^2, \ldots)\). Notice that \(H\) is Polish (e.g., by Corollary 3.39 and Theorem 15.15 in Aliprantis and Border, 2006). Refining \(H\) to \(H^\ell = \{\mu \in H : \forall \ell \geq 1, \text{marg}_{X^{\ell-1}} \mu^{\ell+1} = \mu^\ell\}\) yields the space of coherent belief hierarchies. Brandenburger and Dekel (1993) show the existence of a homeomorphism \(g^\ell : H^\ell \to \Delta(\Theta \times H)\). The space \(H^\ell\) can be further refined in the following manner. Let \(H^0 = H^\ell\) and for every \(k \geq 1\) define \(H^k = \{\mu \in H^{k-1} : g^\ell(\mu) (\Theta \times H^{k-1}) = 1\}\). This yields \(H^{cc} = \cap_{k \geq 1} H^k\), the space of belief hierarchies satisfying common certainty of coherence. Brandenburger and Dekel (1993) also show the existence of a homeomorphism \(g^{cc} : H^{cc} \to \Delta(\Theta \times H^{cc})\). We refer to \(\langle \Theta, (H^{cc})_{i \in I}, (g^{cc})_{i \in I} \rangle\) as the universal \(\Theta\)-based type space. For an analogous yet technically different construction where \(\Theta\) is compact, see Mertens and Zamir (1985); Heifetz (1995) covers the case where \(\Theta\) is possibly non-compact Hausdorff and beliefs are regular Borel probability measures.

**Definition 1** A \(\Theta\)-based type space is a tuple \(\langle \Theta, (T_i)_{i \in I}, (f_i)_{i \in I} \rangle\), where \(\Theta\) and each \(T_i\) are Polish, and each \(f_i : T_i \to \Delta(\Theta \times T_{-i})\) is continuous.

A point in \(T_i\) is denoted by \(t_i\), which is a type for player \(i\). \(f_i(t_i)\) is type \(t_i\)’s posterior on the state of nature and on \(-i\)’s type. Throughout, there is a commonly known common prior \(f \in \Delta(\Theta \times T_i \times T_{-i})\), such that each \(f_i(t_i) () = f(\cdot | t_i)\). A point \((\theta, t_i, t_{-i}) \in \Theta \times T_i \times T_{-i}\) is a state of the world and is the realization of the random \((\tilde{\theta}, \tilde{t}_i, \tilde{t}_{-i})\) distributed according to \(f\).

Each type \(t_i\) in a \(\Theta\)-based type space can be mapped into an infinite hierarchy of beliefs. To see this, let \(h_i^1 : T_i \to \Delta(\Theta \times T_{-i})\) be given by \(h_i^1(t_i) = \text{marg}_{X^0} f_i(t_i)\). \(h_i^1(t_i) \in \Delta(\Theta \times T_{-i})\) is type \(t_i\)’s first-order belief, and corresponds to the marginal on the state of nature of player \(i\)’s posterior. For \(\ell \geq 2\), let \(h_i^\ell : T_i \to \Delta(\Theta \times T_{-i})\) be given by

\[
h_i^\ell(t_i)(E) = f_i(t_i)(\{ (\theta, t_{-i}) \in T_{-i} \times T_i : (\theta, h_i^{\ell-1}(t_{-i}), \ldots, h_i^{\ell-1}(t_{-i})) \in E \})
\]

for every Borel subset \(E \subseteq X^{\ell-1}\). \(h_i^\ell(t_i) \in \Delta(\Theta \times T_{-i})\) is type \(t_i\)’s \(\ell\)-th-order belief. Finally, let \(h_i : T_i \to H\) be given by \(h_i(t_i) = (h_i^1(t_i), h_i^2(t_i), \ldots)\). \(h_i(t_i) \in H\) is type \(t_i\)’s entire belief hierarchy.

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4. For an analogous yet technically different construction where \(\Theta\) is compact, see Mertens and Zamir (1985); Heifetz (1995) covers the case where \(\Theta\) is possibly non-compact Hausdorff and beliefs are regular Borel probability measures.

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We say that $\Theta \times \mathcal{H}^\infty_i \times \mathcal{H}^\infty_{i'},$ where $\mathcal{H}^\infty_i, \mathcal{H}^\infty_{i'} \subseteq H^{cc}$, is a belief-closed subspace of $\Theta \times H^{cc} \times H^{cc}$ if for every $i \in I$ and every $\mu_i \in \mathcal{H}^\infty_i$ we have $g^{cc}(\mu_i) \big( \Theta \times \mathcal{H}^\infty_{i'} \big) = 1$. It can be shown that $h_i(T_i), h_{i'}(T_{i'}) \subseteq H^{cc}$ indeed form a belief-closed subspace of $\Theta \times H^{cc} \times H^{cc}$ (e.g., by Proposition 3 of Battigalli and Siniscalchi, 1999).

**Assumption 1** Each $T_i$ is compact and each $h_i$ is one-to-one.

By construction, $h_i$ is continuous (again, by Proposition 3 of Battigalli and Siniscalchi, 1999). Assumption 1 requires it to be one-to-one. This is a non-redundancy assumption: $h_i(t_i) = h_i(t'_i)$ implies $t_i = t'_i$. If Assumption 1 holds, then each $h_i$ is a continuous one-to-one map from a compact space to the Hausdorff space $H$ (Polish spaces are metrizable, hence Hausdorff). It follows that: (i) each $h_i$ is a closed map; (ii) each $h_i$ is an embedding, i.e., $h_i : T_i \rightarrow h_i(T_i)$ is a homeomorphism (e.g., by Theorem 2.36 in Aliprantis and Border, 2006). Thus, $\Theta \times T_i \times T_{i'}$ is homeomorphic to a belief-closed subspace of $\Theta \times H^{cc} \times H^{cc}$.

### 3 Quotient type spaces

We can now define the $\ell_i$-quotient type spaces. For $\ell_i \geq 1$, let $h_i^{1,\ell_i} : T_i \rightarrow \prod_{i=1}^{\ell_i} \Delta (X^{m-1})$ be given by $h_i^{1,\ell_i}(t_i) = \left(h_i^1(t_i), \ldots, h_i^{\ell_i}(t_i)\right)$. In words, $h_i^{1,\ell_i}(t_i)$ is the partial belief hierarchy of type $t_i$, from $h_i^1(t_i)$ to $h_i^{\ell_i}(t_i)$. It is these partial hierarchies that determine the elements of our quotient spaces.

**Definition 2** The $\ell_i$-quotient type space $(i \in I, \ell_i \in \{1, 2, \ldots\})$ is the space

$$T_i^{\ell_i} = \left\{[t_i] \subseteq T_i : t'_i \in [t_i] \Longleftrightarrow h_i^{1,\ell_i}(t'_i) = h_i^{1,\ell_i}(t_i)\right\} \quad (2)$$

Let $\zeta^{\ell_i} : T_i \rightarrow T_i^{\ell_i}, \zeta^{\ell_i}(t_i) = [t_i] = \left\{t'_i \in T_i : h_i^{1,\ell_i}(t'_i) = h_i^{1,\ell_i}(t_i)\right\}$ be the corresponding quotient map (e.g., see Munkres, 2000). Then we can also write the $\ell_i$-quotient space as $T_i^{\ell_i} = \zeta^{\ell_i}(t_i) : t_i \in T_i \}$. We endow $T_i^{\ell_i}$ with the quotient topology induced by $\zeta^{\ell_i}$: $E \subseteq T_i^{\ell_i}$ is open in $T_i^{\ell_i}$ if and only if $\zeta^{\ell_i}^{-1}(E)$ is open in $T_i$. An element of $T_i^{\ell_i}$ is denoted by $t_i^{\ell_i}$, which we refer to as an $\ell_i$-type and which is an equivalence class of types $t_i \in T_i$.

In many applications, one can find $\ell_i < \infty$ such that $T_i^{\ell_i} = \{t_i : t_i \in T_i \}$. That is, one can find a finite reasoning level $\ell_i$ such that a level-$\ell_i$ player is unboundedly rational (e.g., in the game of Section 4.1.1). Sometimes, however, such a finite $\ell_i$ does not exist (e.g., in the game of Section 3.1.1). It is therefore notionally convenient to identify the unboundedly-rationality benchmark with $\ell_i = \infty$ and let $T_i^{\infty}$ be the quotient space induced by the quotient map $\zeta^{\infty}(t_i) = \{t'_i \in T_i : h_i(t'_i) = h_i(t_i)\}$. For every $i \in I$, we write

$L_i = \left\{\ell_i \in \{1, 2, \ldots\} : T_i^{\ell_i} \neq T_i^{\infty}\right\} \cup \{\infty\}$ for the set of all possible levels.

**Proposition 1** If Assumption 1 holds, then there exists a homeomorphism $\varphi^{\ell_i} : T_i^{\ell_i} \rightarrow h_i^{1,\ell_i}(T_i) (i \in I, \ell_i \in L_i \setminus \{\infty\})$.

**Proof.** Let $\varphi^{\ell_i}(t_i^{\ell_i}) = h_i^{1,\ell_i}(\zeta^{\ell_i}^{-1}(t_i^{\ell_i}))$. Notice that $\varphi^{\ell_i}$ is one-to-one since by definition $\varphi^{\ell_i}(t_i^{\ell_i}) = \varphi^{\ell_i}(t_i'^{\ell_i})$ implies $t_i^{\ell_i} = t_i'^{\ell_i}$. It is continuous since $\zeta^{\ell_i}$ is a quotient map and $h_i^{1,\ell_i} = \varphi^{\ell_i} \circ \zeta^{\ell_i}$ is continuous.
Corollary 1 If Assumption 1 holds, then \( T^\xi_i \) is Polish (\( i \in I, \, \xi_i \in L_i \setminus \{ \infty \} \)).

**Proof.** A subset of a Polish space is Polish if and only if it is a \( G_δ \) set (e.g., by Corollary 3.5, Lemma 3.33, and Alexandrov’s Lemma 3.34 in Aliprantis and Border, 2006). It then suffices to show that \( h^\xi_i : T_i^\xi_i \rightarrow \prod_{m=1}^\xi \Delta (X^{m-1}) \) is closed in \( \prod_{m=1}^\xi \Delta (X^{m-1}) \), as in metrizable spaces every closed set is \( G_δ \) (e.g., by Corollary 3.19 in Aliprantis and Border, 2006). This follows from \( \varphi^\xi_i : T_i^\xi_i \rightarrow \prod_{m=1}^\xi \Delta (X^{m-1}) \) being a closed map. ■

### 3.1 Equilibrium analysis

The defined quotient spaces are compatible with standard equilibrium analysis. Let the space \( A_i \) contain player \( i \)'s pure actions, which are denoted by \( a_i \). A strategy for player \( i \) is a measurable map \( \sigma_i : T_i \rightarrow A_i \). The set of all strategies is denoted by \( \Sigma_i \).

**Definition 3** A strategy \( \sigma_i \in \Sigma_i \) is \( T^\xi_i \)-measurable if there exists a measurable \( \tilde{\sigma}^\xi_i : T_i^\xi_i \rightarrow A_i \) such that

\[
\sigma_i = \tilde{\sigma}^\xi_i \circ \zeta^\xi_i.
\]

In particular, for a \( T^\xi_i \)-measurable strategy \( \sigma_i \) we have that \( \zeta^\xi_i (t_i) = \zeta^\xi_i (t'_i) \) implies \( \sigma_i (t_i) = \sigma_i (t'_i) \) for all \( t_i, t'_i \in T_i \). If player \( i \) is of level \( \xi_i \), then \( \sigma_i \) must be \( T^\xi_i \)-measurable. This way, the corresponding distribution of actions can be derived from the distribution of \( \xi_i \)-types, as if player \( i \)'s type space were \( T^\xi_i \) instead of \( T_i \).

Player \( i \)'s payoff function is given by the measurable \( u_i : \Theta \times A_i \times A_{-i} \rightarrow \mathbb{R} \). A Bayesian game is a tuple \( \Gamma = \langle I, (A_i)_{i \in I}, \Theta, (T_i)_{i \in I}, (f_i)_{i \in I}, (u_i)_{i \in I} \rangle \).

**Definition 4** The strategy profile \( \sigma^* = (\sigma^*_i, \sigma^*_i, \_i) \) is a Bayesian Nash \( \ell \)-equilibrium (\( \ell = (\ell_i, \_i) \in L_i \times L_{-i} \)) of \( \Gamma \) if each \( \sigma^*_i \) is \( T^\ell_i \)-measurable and, for each \( i \in I \) and each \( t_i \in T_i \),

\[
\sigma_i^* (t_i) \in \arg \max_{a_i \in A_i} \mathbb{E} \left[ u_i \left( \tilde{\theta}_{\_i}, a_i, \sigma_i^* (\_i) \right) \bigg| \_i \in \zeta^\ell (t_i) \right] \quad (3)
\]

### 3.1.1 Example: The electronic mail game

Consider the game described in Rubinstein (1989), where \( I = \{ 1, 2 \} \) and \( \Theta = \{ \theta_a, \theta_b \} \) is given by

\[
\begin{array}{cc|cc|cc}
\theta_a = & 1 & 2 & a & b \\
& a & M, M & 1, -L & b \\
& b & -L, 1 & 0, 0 & & \\
\theta_b = & 1 & 2 & a & b \\
& a & 0, 0 & 1, -L & b \\
& b & -L, 1 & M, M & & \\
\end{array}
\]

We leave out mixed strategies because when \( T_i \) is uncountable they involve measurability issues (e.g., see Aumann, 1964, and Milgrom and Weber, 1985) that are beyond the scope of our paper. Indeed, all our examples feature equilibria in pure strategies.

\(^6\)Existence of such a \( \tilde{\sigma}^\xi_i \) implies that \( \sigma_i \) is measurable with respect to the \( \sigma \)-algebra given by the sets \( \left[ \zeta^\xi_i \right]^{-1} (E) \) such that \( E \) is Borel in \( T^\xi_i \).
with $1 < M < L$. The state of nature is $\theta_a$ with probability $\rho \in (0,1)$. Only player 1 is informed about whether $\theta_a$ or $\theta_b$ is the realization of $\tilde{\theta}$. If $\tilde{\theta} = \theta_a$, then an automatic communication protocol has players sequentially sending e-mails to each other - starting from player 1 sending an e-mail to player 2 - that fail to be delivered with probability $\epsilon \in (0,1)$. Let $T_1 = T_2 = \{0, 1, 2, \ldots\}$ count the number of e-mails sent by a given player. By defining $n(t_1, t_2) = |\{t'_1 \in T_1 : t'_1 < t_1\}| + |\{t'_2 \in T_2 : t'_2 < t_2\}| + 1$, we can write the underlying common prior on $\Theta \times T_1 \times T_2$ as

$$f(\theta, t_1, t_2) = \rho \cdot 1_{\{\theta = \theta_a\}} + (1 - \rho)(1 - \epsilon)^{n(t_1, t_2)}\epsilon \cdot 1_{\{\theta = \theta_b\}}$$

for $t_1 - 1 \leq t_2 \leq t_1$ and 0 otherwise. Player $i \in I$ of level $\ell_i \in \{1, 2, \ldots, \infty\}$ maximizes his ex-ante payoff $E_u[\tilde{\theta}, \sigma_1(\tilde{t}_1), \sigma_{-i}(\tilde{t}_{-i})]$ by choosing $\sigma_i : T_i \rightarrow A_i$ subject to $\sigma_i$ being $T_i^\ell$-measurable.\(^7\) In contrast with the unique Bayesian Nash ($\infty, \infty$)-equilibrium, action $b$ may be chosen in a boundedly rational $\ell$-equilibrium.

**Proposition 2** For any $k \in \{1, 2, \ldots\}$, there exists $\ell = (\ell_1, \ell_2) \in \{1, 2, \ldots, \infty\}^2$ and an associated Bayesian Nash $\ell$-equilibrium of the electronic mail game in which players that have sent $k$ or more messages play action $b$.

**Proof.** Let $\ell_1 < \ell_2 \leq \infty$.\(^8\) Notice that $T_1^\ell = T_2^\ell = \overline{\{0\}, \{1, 2, \ldots\}}$ because: (i) only the type that has sent no e-mails believes $\tilde{\theta} = \theta_a$ with positive probability; (ii) all types that have sent at least one e-mail believe $\tilde{\theta} = \theta_b$. Suppose player 1 has sent exactly $\tilde{t}_1 = t_1 \geq 2$ messages. In terms of lower bounds on the other player’s type, player 1 then believes $\tilde{t}_2 \geq t_1 - 1$, that player 2 believes $\tilde{t}_1 \geq t_1 - 1$, that 2 believes 1 believes $\tilde{t}_2 \geq t_1 - 2$, that 2 believes 1 believes 2 believes $\tilde{t}_1 \geq t_1 - 2$, and so forth. Thus, by letting $e(\ell_1) = \max \{t_1 \leq \ell_1 : t_1 = 0 \mod 2\}$ we have $T_1^{\ell_1} = \{\{0\}, \{1\}, \ldots, \{\frac{1}{2} e(\ell_1)\}, \{\frac{1}{2} e(\ell_1) + 1, \ldots\}\}$. Suppose $\ell_2 = t_2 \geq 1$. As lower bounds, player 2 then believes $\tilde{t}_1 \geq t_2$, that 1 believes $\ell_2 \geq t_2 - 1$, that 1 believes 2 believes 1 believes $\tilde{t}_2 \geq t_2 - 2$, and so on. Thus, by letting $o(\ell_2) = \max \{t_2 \leq \ell_2 : t_2 = 1 \mod 2\}$ we have $T_2^{\ell_2} = \{\{0\}, \{1\}, \ldots, \{\frac{1}{2} o(\ell_2) - 1\}, \{\frac{1}{2} o(\ell_2), \ldots\}\}$. It follows that, with $\ell_1 < \ell_2$, $T_2^{\ell_2}$ is no coarser than $T_1^{\ell_1}$.

By iterated elimination of strictly dominated strategies at an interim level, $\sigma_1(0) = a$, which implies $\sigma_2(0) = a$, which implies $\sigma_1(1) = a$, ..., which implies $\sigma_1(\frac{1}{2} e(\ell_1)) = a$, which implies $\sigma_2(\frac{1}{2} e(\ell_1)) = a$. Now suppose $\sigma_1(\frac{1}{2} e(\ell_1) + 1) = \sigma_1(\frac{1}{2} e(\ell_1) + 2) = \ldots = b$. Then $\sigma_2(\frac{1}{2} e(\ell_1) + 1) = \sigma_2(\frac{1}{2} e(\ell_1) + 2) = \ldots = b$. There remains to check that 1’s ex-ante payoff under this strategy is maximal. One shows that the equilibrium is sustained under the condition $\epsilon \leq \epsilon^* = \frac{M-1}{M+L-1}$.

To prove the claim, let $\ell_1$ be such that $\frac{1}{2} e(\ell_1) + 1 = k$. \(\blacksquare\)

This is consistent with experimental evidence by Camerer (2003): subjects tend to play action $b$ already after few messages exchanged.

### 3.2 Countable $\Theta$-based Type Spaces

We say that a $\Theta$-based type space $(\Theta, (T_i)_{i \in I}, (f_i)_{i \in I})$ is **countable** if the following Assumption 2 is verified.

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\(^7\)If $T_i$ is countable, then the ex-ante perspective is equivalent to the interim perspective of Definition 4. If instead $T_i$ is uncountable, a strategy $\sigma_i$ that maximizes the ex-ante payoff need not maximize the interim payoff on a subset of types $t_i$ of null measure.

\(^8\)Equilibria for $\ell_2 < \infty$ and $\ell_2 \leq \ell_1 \leq \infty$ can be found with analogous computations.
Assumption 2 $\Theta \times T_i \times T_{-i}$ is countable.

In this case, $T_i^\ell$ and $T_{-i}^\ell$ can be employed to construct another $\Theta$-based type space. Let $f_i^\ell : T_i^\ell \to \Delta(\Theta \times T_{-i})$ be given by $f_i^\ell(\tilde{t}_i) (\cdot) = \int \cdot [\zeta_i^\ell]^{-1}(t_i^\ell) du$. The product of the identity map on $\Theta$ and the quotient map $\zeta_i^\ell$ defines a map $\zeta_i^\ell : \mathcal{Q}_0(\Theta \times T_{-i}) \to 2^\Delta(\Theta \times T_{-i})$, which in turn induces a $\hat{\zeta}_i^\ell : \Delta(\Theta \times T_{-i}) \to \Delta(\Theta \times T_{-i})$ given by $\hat{\zeta}_i^\ell(g)(E) = g \left( [\zeta_i^\ell]^{-1}(E) \right)$ for $g \in \Delta(\Theta \times T_{-i})$ and $E \subseteq \Theta \times T_{-i}^\ell$. Finally, let $f_i^\ell,\tilde{t}_i = \hat{\zeta}_i^\ell \circ f_i^\ell$.

Proposition 3 If Assumption 2 holds, then $\left\{ \Theta \left( T_i^\ell \right)_{i \in I}, \left( f_i^\ell,\tilde{t}_i \right)_{i \in I} \right\}$ is a countable $\Theta$-based type space $(\ell = (\ell_i, \ell_{-i}) \in L_i \times L_{-i})$.

Proof. $T_i^\ell$ is Polish for $i \in I$, since each of these $\ell_i$-quotient type spaces are countable - possibly finite, as in the e-mail game of Section 3.1.1 - and can be shown to inherit the discrete topology from $T_i$ (i.e., every subset of each $T_i^\ell$ is open). By the discrete topology, each $f_i^\ell$ is continuous. $\hat{\zeta}_i^\ell$ is a continuous map between metrizable spaces, thus $\hat{\zeta}_i^\ell$ is continuous (e.g., by Theorem 15.14 in Aliprantis and Border, 2006). It follows that each composite map $f_i^\ell,\tilde{t}_i$ is continuous. Countability follows from $|\Theta \times T_i^\ell \times T_{-i}^\ell| \leq |\Theta \times T_i \times T_{-i}|$.

Let $\Gamma = \left( I, (A_i)_{i \in I}, \Theta, (T_i)_{i \in I}, (f_i)_{i \in I}, (u_i)_{i \in I} \right)$ and $\Gamma^\ell = \left( I, (A_i)_{i \in I}, \Theta, (T_i^\ell)_{i \in I}, (f_i^\ell,\tilde{t}_i)_{i \in I}, (u_i)_{i \in I} \right)$.

Proposition 4 If Assumption 2 holds, then the strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is a Bayesian Nash $\ell$-equilibrium of $\Gamma$ if and only if the induced strategy profile $\hat{\sigma}^* = (\hat{\sigma}_i^*, \hat{\sigma}_{-i}^*)$ is a Bayesian Nash $(\infty, \infty)$-equilibrium of $\Gamma^\ell$ $(\ell = (\ell_i, \ell_{-i}) \in L_i \times L_{-i})$.

Proof. First, note that $\hat{\sigma}_i^* \left( t_i^\ell \right) = \sigma_i^* \left( [\zeta_i^\ell]^{-1}(t_i^\ell) \right) = \sigma_i^* (t_i)$ for all $t_i \in [\zeta_i^\ell]^{-1}(t_i^\ell)$ because a $T_i^\ell$-measurable strategy is constant over equivalence classes. Take any such $t_i$.

$$E \left[ u_i \left( \hat{\theta}, a_i, \sigma_{-i}^* \left( \tilde{t}_{-i} \right) \right) \mid \tilde{t}_i \in \zeta_i^\ell(t_i) \right] = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} u_i \left( \theta, a_i, \sigma_{-i}^* \left( t_{-i} \right) \right) f \left( \theta, t_{-i} \mid [\zeta_i^\ell]^{-1}(t_i^\ell) \right)$$

$$= \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} u_i \left( \theta, a_i, \hat{\sigma}_{-i}^* \left( t_{-i}^\ell \right) \right) f \left( \left[ \hat{\zeta}_i^\ell \right]^{-1} \left( \theta, t_{-i} \right) \mid [\zeta_i^\ell]^{-1}(t_i^\ell) \right)$$

$$= E \left[ u_i \left( \hat{\theta}, a_i, \sigma_{-i}^* \left( \tilde{t}_{-i}^\ell \right) \right) \mid \tilde{t}_i^\ell \in \hat{\zeta}_i^\ell \right] \left( t_i^\ell \right) \right] \right\}$$

Therefore, an action $a_i$ satisfies (3) for game $\Gamma$ with players’ levels $(\ell_i, \ell_{-i})$ if and only if it satisfies (3) for $\Gamma^\ell$ with players’ levels $(\infty, \infty)$.

4 Uncertainty about players’ levels

More realistically, players’ levels $\ell = (\ell_i, \ell_{-i})$ may be private information. To address this, we treat players of different levels as different players. Such new players have the same information as the original players from whom they are derived (see Definition 5 in Section 4.1 below). As before, let $L_i =$
\[ \{ \ell_i \in \{1, 2, \ldots \} : T^\ell_i \neq T^\infty_i \} \cup \{ \infty \}, i \in I. \]\n
Let \( L = L_i \times L_{-i} \) and assume there is a common prior \( \lambda \in \Delta (L) \) that is common knowledge between the players. Let \( \mathcal{T}_i = \{ \ell_i \in L_i : \ell_i \in \text{supp} (\text{margin}_L, \lambda) \} \). The expanded set of players is \( I^e = \mathcal{T}_i \cup \mathcal{T}_{-i} \). \( |I^e| = \sum_{i \in I} |\text{supp} (\text{margin}_L, \lambda)| \) is the total number of players. The set of players other than \( \ell_i \) in \( I^e_{-\ell_i} = \mathcal{T}_{-i} \) so that, in particular, \( I^e_{-\ell_i} \cap \mathcal{T}_i = \emptyset \) for any \( \ell_i \in I^e \). That goes to say that player \( \ell_i \) does not face any \( \ell_{-i} \), but only some \( \ell_{-i} \).

### 4.1 Equilibrium analysis

Let player \( \ell_i \)'s set of pure actions be \( A^\ell_i = A_i \), with typical element \( a^\ell_i \). A strategy for player \( \ell_i \) is a \( T^\ell_i \)-measurable map \( \sigma^\ell_i \in \Sigma_i \). Player \( \ell_i \)'s posterior on the other player’s level \( \ell_{-i} \) is given by the map \( \lambda_i : \mathcal{T}_i \rightarrow \Delta (L_{-i}), \) such that \( \lambda_i (\ell_i) (\cdot) = \lambda (\cdot | \ell_i) \). Player \( \ell_i \)'s payoff function is \( u_i : \Theta \times A_i \times \left( \prod_{\ell_{-i} \in \mathcal{T}_{-i}} A_{-i} \right) \rightarrow \mathbb{R}, \) where

\[
\begin{align*}
\sigma^\ell_i (t_i) \in \arg \max_{a_i \in A_i} & \mathbb{E} \left[ u_i (\tilde{\theta}, a_i, \sigma_{-i} (\mathcal{T}_{-i})) \bigg| \ell_i \in \zeta^\ell_i (t_i) \right] 
\end{align*}
\]

In words, player \( i \in I \) of level \( \ell_i \) sees player \( i \)'s (-\( i \in I \)) action as a random \( \tilde{\alpha}_{-i} \) with realizations \( \left( a_{-i} \right)_{\ell_{-i} \in \mathcal{T}_{-i}} \) that occur with probabilities \( (\lambda (\ell_i) (\mathcal{T}_{-i}))_{\ell_{-i} \in \mathcal{T}_{-i}} \). The game without uncertainty of Section 3.1.1 is a special case where \( \lambda \) is degenerate at some \( \ell = (\ell_i, \ell_{-i}) \).

It is essential that \( \lambda_i (\ell_i) (\cdot) \) only depends on \( \ell_i \). If \( \lambda_i (\ell_i) (\cdot) \) were to depend on either \( t_i \) or \( t_{-i} \), then \( \sigma^\ell_i \) would depend on them as well, so its argument would not be a point in the product space \( \Theta \times A_i \times \left( \prod_{\ell_{-i} \in \mathcal{T}_{-i}} A_{-i} \right) \). If \( \lambda_i (\ell_i) (\cdot) \) depended on \( \theta \), then player \( i \) could infer the state of nature by knowing his level. He would therefore be conditioning his expected payoff both on his type belonging to the respective equivalence class and on his level being \( \ell_i \). If this were true, then \( -i \) would need to form beliefs about \( i \)'s level of sophistication, but this is precisely what we want to avoid.

The function \( u_i \) has to be measurable. A sufficient condition for this is its continuity - which is satisfied in our applications.

**Definition 5** The strategy profile \( \sigma^* = \left( (\sigma^i_{\ell_i})_{\ell_i \in \mathcal{T}_i}, (\sigma_{-i}^* )_{\ell_{-i} \in \mathcal{T}_{-i}} \right) \) is a Bayesian Nash \( \lambda \)-equilibrium \( (\lambda \in \Delta (L)) \) of \( \Gamma \) if each \( \sigma^i_{\ell_i} \) is \( T^\ell_i \)-measurable and, for each \( \ell_i \in I^e \) and each \( t_i \in T_i \),

\[
\begin{align*}
\sigma^i_{\ell_i} (t_i) \in \arg \max_{a_i \in A_i} & \mathbb{E} \left[ u_i (\tilde{\theta}, a_i, \sigma_{-i} (\mathcal{T}_{-i})) \bigg| \ell_i \in \zeta^i (t_i) \right] 
\end{align*}
\]

### 4.1.1 Example: A game with private and public information

Consider a version of the game described in Morris and Shin (2002), where \( I = \{1, 2\} \) and the state of nature \( \theta \) follows an improper uniform distribution on \( \Theta = \mathbb{R} \). Both players receive a public signal \( y = \theta + \epsilon_y \), where \( \epsilon_y \sim U [-c, c], c > 0 \). In addition, each of them privately observes a signal \( x_i = \theta + \epsilon_i \), where \( \epsilon_i \sim U [-c, c] \), \( i \in I \). The random variables \( \theta, \epsilon_y, \epsilon_i \), and \( \epsilon_q \) are all independent of each other. Payoffs are given by

\[
u \in (0, 1) \text{ is a parameter. We have } T_2 = \{ t_i = (x_i, y) : (x_i, y) \in \mathbb{R}^2 \} \text{ and } T^\ell_i = \{ \zeta^i (t_i) : t_i \in T_i \}. \text{ In particular: (i) } \zeta^i (t_i) \text{ is such that } \zeta^i_1 (t_i) = \zeta^i_2 (t_i) \text{ if and only if } [x_i = x'_i \text{ and } y = y'] \text{ or } [x_i = y' \text{ and } y = x'_i];\]
\(\zeta_i^2 = \zeta_i^3 = \ldots = \zeta_i^\infty.\) To interpret, a player of level 1 does not distinguish private and public signal. Each player \(i\) can be either of level 1 or \(\infty\). The common prior on levels \(\lambda \in \Delta \left( (1,\infty)^2 \right)\) is given by 
\[
\lambda(1,1) = p^2, \lambda(1,\infty) = \lambda(\infty,1) = p(1-p), \text{ and } \lambda(\infty,\infty) = (1-p)^2.
\]

The payoff function for player \(i\) of level \(\ell_i \in \{1,\infty\}\) is
\[
u_i \left( \theta_i, a_{i,1}, a_{i,\infty} \right) = - (1 - \nu) \left( a_{i,1} - \theta \right)^2 - \nu p \left( a_{i,1} - a_{i,\infty} \right)^2 - \nu (1-p) \left( a_{i,1} - a_{i,\infty} \right)^2 \tag{9}\]

\(\nu\) We look for symmetric linear equilibria of the form \(a_i = \kappa_i x_i + (1 - \kappa_i) y\), \(\kappa_i \in (0,1), \ell \in \{1,\infty\}, i \in I\).

Proposition 5 For each \(p \in (0,1)\) there exists a Bayesian Nash \(\lambda\)-equilibrium such that the mean weight put on the private signal (i.e., \(p \kappa_i + (1-p) \kappa_\infty\)) is larger than when there is common certainty that both players are unboundedly rational (i.e., when \(p = 0\)).

Proof. Let \(\chi_i = \min \{x_i, y\}\) and \(\chi_i = \max \{x_i, y\}\). The first-order condition for a level-\(\ell_i\) player \(i\) is
\[
u_i = (1 - \nu) E_i^\ell \theta + \nu p E_i^\ell a_{i,1}^{\ell_i} + \nu (1-p) E_i^\ell a_{i,\infty}^{\ell_i}, \text{ where for any random variable } z, \ E_i^\ell z = E \left[ z \mid \zeta_i^\ell \left( t_i \right) \right].
\]

Notice that \(E_i^\ell z = E \left[ z \mid \chi_i, \chi_i \right]\) and \(E_i^\infty z = E [z \mid x_i, y_i]\). If \(\kappa_1 = \frac{1}{2}\), then \(a_i = (1 - \nu) E_i^\ell \theta + \nu p E_i^\ell \left[ \frac{1}{2} x_i + \frac{1}{2} y \right] + \nu (1-p) E_i^\ell \left[ \kappa_\infty x_i + (1 - \kappa_\infty) y \right]\). From \(P \left[ \chi_i = y \right] = 1 - P \left[ \chi_i = x \right] = \frac{1}{2}\) and \(E_i^\ell x_i = E_i^\ell \theta = \frac{1}{2} \chi_i + \frac{1}{2} \chi_i\), we have that for any \(\alpha \in (0,1)\)
\[
E_i^\ell [\alpha y + (1 - \alpha) x_i] = \frac{1}{2} \left[ \alpha \chi_i + (1 - \alpha) \frac{\chi_i + \chi_i}{2} \right] + \frac{1}{2} \left[ \alpha \chi_i + (1 - \alpha) \frac{\chi_i + \chi_i}{2} \right] = \frac{1}{2} \chi_i + \frac{1}{2} \chi_i,
\]

The best-responses to \(\kappa_1 = \frac{1}{2}\) and any \(\kappa_\infty \in (0,1)\) is therefore \(\kappa_1 = \frac{1}{2}\). As to \(\kappa_\infty\), we have \(a_i = (1 - \nu) E_i^\infty \theta + \nu p E_i^\infty \left[ \frac{1}{2} x_i + \frac{1}{2} y \right] + \nu (1-p) E_i^\infty \left[ \kappa_\infty x_i + (1 - \kappa_\infty) y \right]\), from which \(\kappa_\infty = \frac{2 - 2p + p^2}{2(2 - p + p^2)}\).

Since \(p \in (0,1)\), it is \(p \kappa_i + (1-p) \kappa_\infty \in \left( \frac{1 - \nu}{2 - \nu + \nu p}, \frac{1}{2} \right]\), which is larger than the standard \((\infty, \infty)\)-equilibrium weight \(\frac{1 - \nu}{2 - \nu + \nu p}\). □

These results are consistent with experimental evidence by Cornand and Heinemann (2009). Also, notice that
\[
\lim_{p \to 1} \kappa_\infty = \frac{1}{2}, \text{ which is the best-response of a level-2 player to a level-1 player who equally randomizes between signals in their level-k/cognitive hierarchy model (Camerer, Ho, and Chong, 2004, Nagel, 1995, Stahl and Wilson, 1994, 1995).}

5 Related Literature

Kets (2010) characterizes extended type structures, which feature Harsanyi type structures as special cases. As in the Harsanyi framework, each type in an extended type structure is associated with beliefs about the state of nature and the other player’s types, yet the beliefs of different types are allowed to be defined on different \(\sigma\)-algebras. These \(\sigma\)-algebras reflect types’ coarseness of perception of their opponent’s high-order beliefs. The author shows that standard Harsanyi type structures are characterized by common certainty that each player has an infinite depth of reasoning. The difference with our model is twofold: (i) we do not impose cognitive limitations on player \(i\)’s capacity to perceive player \(i\)’s high-order beliefs, but rather on

\(\footnote{If } y \neq y', \text{ then types } (x_i, y) \text{ and } (x_i', y') \text{ such that } [x_i = y' \text{ and } y = x_i] \text{ have different second-order beliefs.}\)
i’s capacity to perceive his own high-order beliefs; (ii) we place emphasis on being in a position to retain
the standard game-theoretical tools for equilibrium analysis of incomplete-information games. By contrast,
in Kets’ (2010) analysis of the e-mail game, player \( i \) with depth of reasoning \( \ell_i \) plays strategies that are
rationalizable for him given his coarse perception of player \( \neg i \)’s high-order beliefs. This amounts to each
player \( i \) perceiving a different game of incomplete information in which: (i) his set of types is the same as
in the original game; (ii) player \( \neg i \)’s set of types is changed according to \( i \)’s own perception of player \( \neg i \)’s
high-order beliefs; (iii) player \( i \) plays strategies that are rationalizable in the modified game.\(^{10}\)

Jehiel (2005) proposes a solution concept for multi-stage games with perfect information, the *analogy-based
expectation equilibrium*. Jehiel and Koessler (2008) make use of analogy-based expectations in static two-
player games of incomplete information. The basic idea behind their analogy-based expectation equilibrium is
that player \( i \) plays best-responses to player \( \neg i \)’s average strategy within analogy classes, which are bundles
of states of the world. The authors interpret the analogy-based expectation equilibrium as “the limiting
outcome of a learning process involving populations of players \( i = 1, 2 \), who would get a coarse feedback
about the past behavior of players in population \( j \neq i \) and no feedback on their own past performance
until they exit the system” (p. 534). They then apply their theory to the e-mail game and find the same
threshold for the probability with which each message gets lost as we do. However, this should be seen as
a mere coincidence, as generally analogy-based expectation equilibria need not coincide with Bayesian Nash
equilibria of a game with modified information partitions.

Dulleck (2007) resolves the e-mail game paradox by assuming that a player loses track of the number of
e-mails received in a given interval and that the corresponding information structure is common knowledge
between players. While this is close in spirit to our approach, the analysis falls short of addressing formally
where the confusion about the e-mail count originates, and the model is specific to the e-mail game, leaving
open how it can be adapted to other situations of strategic interaction.

Finally, Strzalecki (2010) develops a level-\( k \)/cognitive hierarchy model to show that coordination on the
Pareto-efficient outcome is possible in the e-mail game, provided sophisticated players put enough weight
on the other being less sophisticated. The model builds on a non-equilibrium approach according to which
players are boundedly rational in terms of reasoning about actions (i.e., what \( i \) thinks \( \neg i \) thinks \( \ldots \) will
play) but unboundedly rational in terms of epistemic reasoning (i.e., what \( i \) thinks \( \neg i \) thinks \( \neg i \) thinks \( \ldots \) about
the state of nature). On the contrary, we do not depart from equilibrium analysis and our quotient type spaces
have clear-cut interpretations of players’ ability – or inability – to precisely perceive high-order beliefs about
states of nature.

References


\(^{10}\)In Kets’ (2010) paper, iterated elimination of strictly dominated strategies at an interim level also takes into account
uncertainty about players' depth of reasoning. This is to say that since an action for type \( t_i \) may or may not be a best-response
depending on how sophisticated \( i \) is, an action for type \( t_{\neg i} \) may or may not be eliminated depending on how sophisticated \( \neg i \)
thinks \( i \) is.


