Equivalence of piecewise-linear approximation and Lagrangian relaxation for network revenue management

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Abstract

The network revenue management (RM) problem arises in airline, hotel, media, and other industries where the sale products use multiple resources. It can be formulated as a stochastic dynamic program but the dynamic program is computationally intractable because of an exponentially large state space, and a number of heuristics have been proposed to approximate it. Notable amongst these—both for their revenue performance, as well as their theoretically sound basis—are approximate dynamic programming methods that approximate the value function by basis functions (both affine functions as well as piecewise-linear functions have been proposed for network RM) and decomposition methods that relax the constraints of the dynamic program to solve simpler dynamic programs (such as the Lagrangian relaxation methods). In this paper we show that these two seemingly distinct approaches coincide for the network RM dynamic program, i.e., the piecewise-linear approximation method and the Lagrangian relaxation method are one and the same.

Key words. network revenue management, linear programming, approximate dynamic programming, Lagrangian relaxation methods.

Revenue management is the control of the sale of a limited quantity of a resource (hotel rooms for a night, airline seats, advertising slots etc.) to a heterogenous population with different valuations for a unit of the resource. The resource is perishable, and for simplicity sake, we assume that it perishes at a fixed point of time in the future. Sale is online, so the firm has to decide what products to offer (at a given price for each product), the tradeoff being selling too much at too low a price early and running out of capacity, or, rejecting too many low-valuation customers and ending up with excess unsold inventory.

In industries such as hotels and airlines the products consume bundles of different resources (multi-night stays, multi-leg itineraries) and the decision to accept or reject a particular product at a certain price depends on the future demands and revenues for all the

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resources used by the product and indirectly, on all the resources in the network. Network revenue management (network RM) is control based on the demands for the entire network. Chapter 3 of Talluri and van Ryzin [7] contains all the necessary background on network RM.

The network revenue management problem can be formulated as a stochastic dynamic program, but computing the value function becomes intractable due to the high dimensionality of the state space. As a result, researchers have focussed on developing approximation methods. Notable amongst these—both for their revenue performance, as well as their theoretically sound basis—are approximate dynamic programming methods that approximate the value function by basis functions (both affine functions as well as piecewise-linear functions have been proposed for network RM) and decomposition methods that relax the constraints of the dynamic program to solve simpler dynamic programs (such as the Lagrangian methods). In this paper we show that these two seemingly distinct approaches coincide for the network revenue management dynamic program. Specifically, we show that the piecewise-linear approximation method and the Lagrangian relaxation method are one and the same.

As a by-product, we derive some auxiliary results of independent interest: (i) we give a polynomial-time separation procedure for the piecewise-linear approximation linear program, and (ii) we show that the optimal solution of the piecewise-linear approximation satisfies monotonicity conditions similar to that of a single-resource dynamic program.

The rest of the paper is organized as follows. In §1 we give a brief survey of the relevant literature, in §2 we formulate the network revenue management problem as a dynamic program. In §3 we describe the approximate dynamic programming approach with piecewise-linear basis functions, in §4 we describe the Lagrangian relaxation approach, and in §5 we show that the two approaches are equivalent; §6 concludes.

1 Relevant literature

Approximate dynamic programming (DP) is the name generally given for methods that replace the value function of a (difficult) dynamic program with basis functions and solve the simplified problem as an approximation. In this stream of literature, the linear programming approach consists of formulating the dynamic program as a linear program with state-dependent variables representing the value functions and then replacing them by particular functional forms to find the best approximation within that class of functions. In the network RM context, this approach was first investigated by Adelman [1] who uses affine functions (Zhang and Adelman [10] extend this to the choice model of network RM). A natural extension is to study piecewise-linear functions as they are very flexible and indeed, for the single-resource dynamic program, optimal. In this vein, Farias and van Roy [2] propose a piecewise-linear approximation with concavity constraints and investigate its performance for network RM using constraint sampling as the resulting linear program has an exponential number of constraints and cannot be solved easily. Meissner and Strauss [5] extend this to the choice model of network RM using aggregation over states to reduce the number of variables.

Another stream of literature revolves around the Lagrangian relaxation approach to dynamic programming. Here the idea is to relax certain constraints in the dynamic program by associating Lagrange multipliers with them so that the problem decomposes into simpler problems. For network RM, Topaloglu [9] and Kunnumkal and Topaloglu [4] take this approach. Computational results from Topaloglu [9] indicate that Lagrangian relaxation with product-specific Lagrange multipliers gives consistent and clearly superior revenues compared to the other methods (including the affine relaxation of Adelman [1], but the piecewise-linear relaxation was not included, perhaps because it was not known how to solve it exactly).

How do these seemingly different approaches compare with each other? In a recent paper, Tong and Topaloglu [8] establish the equivalence between the affine relaxation of Adelman [1] and the Lagrangian relaxation of Kunnumkal and Topaloglu [4]. In this paper we show that the piecewise-linear approximation (investigated in Farias and van Roy [2], but without needing their concavity constraints explicitly) and the product and time-specific Lagrangian relaxation of Topaloglu [9] coincide, that is, they in fact represent the same linear program.

2 Problem Formulation

We consider a network revenue management problem with a set of $\mathcal{I} = \{1, \dots, m\}$ resources (for example flight legs on an airline network), $\mathcal{J} = \{1, \dots, n\}$ products (for example itinerary-fare combinations) that use the resources in $\mathcal{I}$ at the end of τ time periods (booking horizon), with time being indexed from 1 to τ. We assume that each product uses at most one unit of each resource, and if j uses resource i, we represent it as $i \in j$, and all j that use resource i by $\{j | j \ni i\}$. Throughout, we index resources by i, products by j and time periods by t. We use $\mathbb{1}_{[\cdot]}$ as the indicator function, 1 if true and 0 if false.

Booking requests for products come in over time and we let p_{jt} denote the probability that we get a request for product j at time period t. We make the standard assumption that the time periods are small enough so that we get a request for at most one product in each time period. Throughout we assume that $p_{jt} > 0$ for all j. Note that this is without loss of generality because if $p_{jt} = 0$ for some product j, then we can simply discard that product and optimize over a smaller number of products. We also assume that $\sum_j p_{jt} = 1$ for all time periods t. This is also without loss of generality because we can add a dummy product with negligible revenue on each resource. We let f_j denote the revenue associated with product j.

Given a request for product j, the airline has to decide online whether to accept or reject the request. In making this decision, the airline is constrained by the available capacities on the resources. That is, it can accept a booking request only if there is sufficient capacity on all the resources that the product uses. An accepted request generates revenue and consumes capacity on the resources used by the product; a rejected request does not generate any revenue and simply leaves the reservation system.

Let $\vec{r}_t = [r_{it}]$ be the m-dimensional vector representing the capacity on the resources at time period t, so that $\vec{r_1} = [r_{i1}]$ denotes the initial capacity on the resources. We let $u_{jt} \in \{0, 1\}$ indicate the control—1 if we accept product j at time t and 0 otherwise, and let $\vec{u}_t = [u_{jt}]$ be the n-dimensional 0-1 vector representing the acceptance decisions. The set of controls $\vec{u}_t$ have to be feasible, that is, if there is no capacity on any resource that j uses, then u_{jt} should be zero. We represent this by $\vec{u}_t \in \mathcal{U}(\vec{r}_t)$ where $\mathcal{U}(\vec{r}) = \{\vec{u} \in \{0, 1\}^n \,|\, u_j \leq r_i \,\forall j, i \in j\}$, the set of *acceptable* products for each state.

The value functions $V_t(\cdot)$ can be obtained through the optimality equations

$$V_t(\vec{r}_t) = \max_{\vec{u}_t \in \mathcal{U}(\vec{r}_t)} \sum_j p_{jt} u_{jt} [f_j + V_{t+1}(\vec{r}_t - \sum_{i \in j} \vec{e}_i) - V_{t+1}(\vec{r}_t)] + V_{t+1}(\vec{r}_t),$$

where $\vec{e}_i$ is a vector with a one in the ith component and zeroes elsewhere, and the boundary condition is $V_{\tau+1}(\cdot) = 0$. $V_1(\vec{r}_1)$ gives the optimal expected total revenue over the booking horizon.

The value functions can, alternatively, be obtained by solving the linear program

$$\begin{aligned}
&\min_{V_t(\cdot)} && V_1(\vec{r}_1) \\
&\text{s.t} && \\
&(DP_{LP}) && V_t(\vec{r}) \geq \sum_j p_{jt} u_{jt} [f_j + V_{t+1}(\vec{r} - \sum_{i \in j} \vec{e}_i) - V_{t+1}(\vec{r})] + V_{t+1}(\vec{r}) \\
& && \qquad\qquad\qquad\qquad \forall t, \vec{r} \in \mathcal{R}, \vec{u}_t \in \mathcal{U}(\vec{r}) \\
& && V_{\tau+1}(\cdot) = 0, V_t(\cdot) \geq 0,
\end{aligned}$$

where $\mathcal{R} = \{\vec{x} \,|\, x_i \in \{0, \ldots, r_{i1}\} \forall i\}$ (Adelman [1]). Note that the decision variables in the above linear program are $\{V_t(\vec{r}) \,|\, \forall t, \vec{r} \in \mathcal{R}\}$ and that $|\mathcal{R}|$ is exponential in the initial capacity on the resources. Computing the value functions either through the optimality equations or the linear program quickly becomes intractable. In the following sections, we describe two approximation methods.

3 Piecewise-Linear Approximation

We use a separable, piecewise-linear approximation to the value function in (DP_{LP}). That is, we approximate the value function[1]

$$V_t(\vec{r}) \approx \sum_i v_{it}(r_i), \,\forall \vec{r} \in \mathcal{R}.$$

[1]Adelman [1] uses the *affine* relaxation $V_t(\vec{r}) \approx \theta_t + \sum_i r_i v_{it}$ but we do not need the offset term θ_t for piecewise-linear approximations as we can use the transformation $\bar{v}_{it}(r_i) = \theta_t/m + v_{it}(r_i)$.

Substituting this approximation into (DP_{LP})

$$
\begin{aligned}
V^{PL} = \min_{v} \quad & \textstyle\sum_i v_{i1}(r_{i1}) && (1)\\
\text{s.t} \quad & && \\
(PL) \quad \textstyle\sum_i v_{it}(r_i) \geq \; & \sum_j p_{jt}u_{jt}[f_j + \sum_{i\in j} v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] && (2)\\
& + \sum_i v_{i,t+1}(r_i) \quad \forall t, \vec{r} \in \mathcal{R}, \vec{u}_t \in \mathcal{U}(\vec{r}) && \\
& v_{i,\tau+1}(\cdot) = 0, v_{it}(\cdot) \geq 0, &&
\end{aligned}
$$

where the decision variables are $\{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ and $\mathcal{R}_i = \{0, \dots, r_{i1}\}$. The number of decision variables in (PL) is $\sum_i r_{i1}\tau$ which is manageable. However, since (PL) has an exponential number of constraints of type (2), we need to use a separation algorithm to generate constraints on the fly to solve (PL) (Grötschel, Lovász, and Schrijver [3]). In §5, we show that the separation can be carried out efficiently for (PL).

Note that we have not imposed any conditions on the slopes $v_{it}(r_i) - v_{it}(r_i - 1)$. Lemma 1 describes some properties of the optimal solutions of (PL). In particular, Lemma 1 says constraints of the form $v_{it}(r_i) - v_{it}(r_i - 1) \geq v_{it}(r_i + 1) - v_{it}(r_i)$ are redundant for (PL). That is, the optimal objective function value is not changed by adding these constraints to (PL).

Lemma 1. *There exists an optimal solution $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ to (PL) such that*
(i) $\hat{v}_{it}(r_i) - \hat{v}_{it}(r_i - 1) \geq \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1)$ for all t, i and $r_i \in \mathcal{R}_i$,
(ii) $\hat{v}_{it}(r_i) - \hat{v}_{it}(r_i - 1) \geq \hat{v}_{it}(r_i + 1) - \hat{v}_{it}(r_i)$ for all $t, i,$ and $r_i \in \mathcal{R}_i$,
where we define $\hat{v}_{it}(-1) = -\infty$ and $\hat{v}_{it}(r_{i1} + 1) = \hat{v}_{it}(r_{i1})$ for all t and i.

Proof. Appendix. □

The following lemma states that there exists an optimal solution to (PL) such that constraint (2) is satisfied as an equality for all time periods t for at least one state $\vec{r}$.

Lemma 2. *There exists an optimal solution to (PL), $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$, such that for every time period t we have*

$$\max_{\vec{r}\in\mathcal{R}, \vec{u}_t\in\mathcal{U}(\vec{r})} \left\{ \sum_j p_{jt}u_{jt}[f_j + \sum_{i\in j} \hat{v}_{i,t+1}(r_i - 1) - \hat{v}_{i,t+1}(r_i)] + \sum_i \hat{v}_{i,t+1}(r_i) - \sum_i \hat{v}_{it}(r_i) \right\} = 0.$$

Proof. Appendix. □

4 Lagrangian Relaxation

Topaloglu [9] proposes a Lagrangian relaxation approach that decomposes the network revenue management problem into a number of single-resource problems by decoupling the acceptance decisions for a product over the resources that it uses via product and time-specific Lagrange multipliers.

Let $\lambda = \{\lambda_{ijt} \,|\, \forall t, j, i \in j\}$ denote the Lagrange multipliers and

$$\mathcal{U}_i(r_i) = \{\vec{u}_i \in \{0,1\}^n \,|\, u_{ij} \leq r_i, \, \forall j \ni i\}.$$

We solve the optimality equation

$$\vartheta_{it}^{\lambda}(r_{it}) = \max_{\vec{u}_{it} \in \mathcal{U}_i(r_{it})} \sum_{j \ni i} p_{jt} u_{ijt} [\lambda_{ijt} + \vartheta_{i,t+1}^{\lambda}(r_{it} - 1) - \vartheta_{i,t+1}^{\lambda}(r_{it})] + \vartheta_{i,t+1}^{\lambda}(r_{it})$$

for resource i, with the boundary condition $\vartheta_{i,\tau+1}^{\lambda}(\cdot) = 0$.

It is possible to show that

$$V_t^{\lambda}(\vec{r}_t) = \sum_j p_{jt} [f_j - \sum_{i \in j} \lambda_{ijt}]^+ + \sum_i \vartheta_{it}^{\lambda}(r_{it}) \tag{3}$$

is an upper bound on $V_t(\vec{r}_t)$, where we use $[x]^+ = \max\{x, 0\}$ (see Topaloglu [9]). We find the tightest upper bound on the optimal expected revenue by solving

$$V^{LR} = \min_{\lambda} V_1^{\lambda}(\vec{r}_1).$$

Talluri [6] shows that the optimal Lagrange multipliers satisfy $\sum_{i \in j} \lambda_{ijt} = f_j$ for all j and t.

Proposition 1. *There exists $\hat{\lambda} \in \arg\min_{\lambda} V_1^{\lambda}(\vec{r}_1)$ that satisfy $\hat{\lambda} \geq 0$ and $\sum_{i \in j} \hat{\lambda}_{ijt} = f_j$ for all j and t.*

Proof. Appendix. □

Proposition 1 implies that we can find the optimal Lagrange multipliers by solving $V^{LR} = \min \sum_i \vartheta_{i1}^{\lambda}(r_{i1})$ subject to $\sum_{i \in j} \lambda_{ijt} = f_j, \lambda_{ijt} \geq 0$, for all $t, j, i \in j$. Note that $\vartheta_{i1}^{\lambda}(r_{i1})$ is the value function of a single-resource revenue management problem with revenues $\{\lambda_{ijt} \,|\, \forall j \ni i, t\}$ on resource i. Therefore, we can also obtain the optimal Lagrange multipliers through the linear programming formulation of the dynamic program for the revenue management problem on a single resource:

$$V^{LR} = \min_{\lambda, \nu} \quad \sum_i \nu_{i1}(r_{i1}) \tag{4}$$

$$\text{s.t.}$$

$$(LR) \quad \nu_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_{ijt} [\lambda_{ijt} + \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)]$$

$$+\nu_{i,t+1}(r_i) \quad \forall t, i, r_i \in \mathcal{R}_i, \vec{u}_{it} \in \mathcal{U}_i(r_i) \tag{5}$$

$$\sum_{i \in j} \lambda_{ijt} = f_j \quad \forall t, j \tag{6}$$

$$\lambda_{ijt} \geq 0, \nu_{it}(\cdot) \geq 0, \nu_{i,\tau+1}(\cdot) = 0.$$

5 Equivalence of the Piecewise-Linear Approximation and the Lagrangian Relaxation Approaches

In this section we show that the piecewise-linear approximation and the Lagrangian relaxation approaches are equivalent, in that they yield the same upper bound on the value function. This also shows that the Lagrangian relaxation approach yields the tightest separable, piecewise-linear value function approximation to the network RM dynamic program.

Proposition 2. $V^{PL} = V^{LR}$.

In §5.1 we show that constraints (2) in (PL) can be separated by solving a linear program. We use this result to prove Proposition 2 in §5.2. We describe a polynomial-time separation algorithm for (PL) in §5.3.

5.1 Separation for (PL)

Since (PL) has an exponential number of constraints of type (2), we use a separation algorithm to solve (PL) (equivalence of efficient separation and solvability of a linear program is due to the well-known work of Grötschel et al. [3]). Constraint generation involves solving the following separation problem: Prove that a given set of values $\bar{v} = \{\bar{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ satisfy constraint (2) for all t, $\vec{r} \in \mathcal{R}$ and $\vec{u}_t \in \mathcal{U}(\vec{r})$, and if not, find a violated constraint. Throughout we assume that $\bar{v}$ satisfies $\bar{v}_{it}(r_i) - \bar{v}_{it}(r_i - 1) \geq \bar{v}_{it}(r_i + 1) - \bar{v}_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$. This is without loss of generality, since by Lemma 1, we can add these constraints to (PL) without affecting its optimal objective function value.

Let $\Delta_{it}(r_i) = \bar{v}_{i,t+1}(r_i) - \bar{v}_{it}(r_i)$ and $\psi_{it}(r_i) = \bar{v}_{it}(r_i) - \bar{v}_{it}(r_i - 1)$ for $r_i \in \mathcal{R}_i$. Note that since $\bar{v}_{it}(-1)$ is defined to be $-\infty$, we have $\psi_{it}(0) = \infty$. Note also that $\psi_{it}(r_i)$ is nonincreasing in r_i and that since $\psi_{it}(r_{i1}) = \bar{v}_{it}(r_{i1}) - \bar{v}_{it}(r_{i1} - 1) \geq \bar{v}_{it}(r_{i1} + 1) - \bar{v}_{it}(r_{i1}) = 0$, we have $\psi_{it}(r_i) \geq 0$ for all $r_i \in \mathcal{R}_i$. Let

$$\Phi_t(\bar{v}) = \max_{\vec{r} \in \mathcal{R}, \vec{u}_t \in \mathcal{U}(\vec{r})} \sum_j p_{jt} u_{jt} [f_j - \sum_{i \in j} \psi_{i,t+1}(r_i)] + \sum_i \Delta_{it}(r_i). \tag{7}$$

The separation problem for a set of values $\bar{v}$ is resolved by obtaining the value of $\Phi_t(\bar{v})$ and checking if for any t, $\Phi_t(\bar{v}) > 0$.

We show that the above optimization problem can be solved efficiently as a linear program. This result is useful for two reasons. First, it helps us in establishing the equivalence between the piecewise linear approximation and the Lagrangian relaxation approaches. Secondly, it shows that separation can be efficiently carried out for (PL).

For each time period t, we split the revenue of product j, f_j, among the resources that it consumes using variables λ_{ijt}. The variable λ_{ijt} represents the revenue allocated to resource $i \in j$ and satisfies $\sum_{i \in j} \lambda_{ijt} = f_j$ and $\lambda_{ijt} \geq 0$ for all $i \in j$. Letting $\lambda = \{\lambda_{ijt} \,|\, \forall t, j, i \in j\}$, we solve the problem

$$\Pi^{\lambda}_{it}(\bar{v}) = \max_{r_i \in \mathcal{R}_i, \vec{u}_{it} \in \mathcal{U}_i(r_i)} \sum_{j \ni i} p_{jt} u_{ijt} [\lambda_{ijt} - \psi_{i,t+1}(r_i)] + \Delta_{it}(r_i) \tag{8}$$

for each resource i. The following lemma states that $\sum_i \Pi_{it}^{\lambda}(\bar{v})$ is an upper bound on $\Phi_t(\bar{v})$.

Lemma 3. *If λ satisfies $\sum_{i\in j} \lambda_{ijt} = f_j$ and $\lambda_{ijt} \geq 0$ for all t, j and $i \in j$, then $\Phi_t(\bar{v}) \leq \sum_i \Pi_{it}^{\lambda}(\bar{v})$.*

Proof. If $\left(\vec{r} = [r_i], \vec{u}_t \in \mathcal{U}(\vec{r})\right)$ is optimal for problem (7), then $\vec{u}_t \in \mathcal{U}_i(r_i)$ and consequently $(r_i, \vec{u}_t)$ is feasible for problem (8). □

We next show that the upper bound is tight. That is, letting

$$\Pi_t(\bar{v}) = \min_{\{\lambda \mid \sum_{i\in j} \lambda_{ijt} = f_j \, \forall j; \lambda_{ijt} \geq 0 \, \forall j, i \in j\}} \sum_i \Pi_{it}^{\lambda}(\bar{v}) \tag{9}$$

we have $\Phi_t(\bar{v}) = \Pi_t(\bar{v})$.

First, we show that problem (8) can be written as the following linear program

$$\begin{aligned} \Pi_{it}^{\lambda}(\bar{v}) = \min_{w,z} \quad & w_{it} && (10) \\ \text{s.t.} \quad & && \\ (S_i) \quad & w_{it} \geq \sum_{j \ni i} z_{ijtr} + \Delta_{it}(r) \quad \forall r \in \mathcal{R}_i && (11) \\ & z_{ijtr} \geq p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)] \quad \forall j \ni i, r \in \mathcal{R}_i && (12) \\ & z_{ijtr} \geq 0 \, \forall j \ni i, r \in \mathcal{R}_i. && \end{aligned}$$

Lemma 4. *The linear program (S_i) is equivalent to (8).*

Proof. Appendix. □

We can, therefore, formulate problem (9) as the linear program

$$\begin{aligned} \Pi_t(\bar{v}) = \min_{\lambda,w,z} \quad & \sum_i w_{it} && (13) \\ \text{s.t.} \quad & && \\ (S) \quad & w_{it} \geq \sum_{j \ni i} z_{ijtr} + \Delta_{it}(r) \quad \forall i, r \in \mathcal{R}_i && (14) \\ & z_{ijtr} \geq p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)] \quad \forall i, j \ni i, r \in \mathcal{R}_i && (15) \\ & \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall j && (16) \\ & \lambda_{ijt} \geq 0 \quad \forall i, j \ni i && (17) \\ & z_{ijtr} \geq 0 \quad \forall i, j \ni i, r \in \mathcal{R}_i. && (18) \end{aligned}$$

Let $\xi_{it}(r) = w_{it} - [\sum_{j \ni i} z_{ijtr} + \Delta_{it}(r)]$ denote the slack in constraint (14), let

$$B_i(\lambda, w, z) = \{r \in \mathcal{R}_i \,|\, \xi_{it}(r) = 0\}$$

denote the set of binding constraints of type (14) and let $B_i^c(\lambda, w, z)$ denote its complement. Note that if $(\hat{\lambda}, \hat{w}, \hat{z})$ is an optimal solution, then $B_i(\hat{\lambda}, \hat{w}, \hat{z})$ is nonempty, since for each resource i, there exists some $\hat{r}_i \in \mathcal{R}_i$ such that constraint (14) is satisfied as an equality. The following proposition is a key result.

Proposition 3. *There exists an optimal solution $(\hat{\lambda}, \hat{w}, \hat{z})$ to (S) and $\{\hat{r}_i \,|\, \hat{r}_i \in B_i(\hat{\lambda}, \hat{w}, \hat{z}), \forall i\}$ such that for each j, we either have $\hat{\lambda}_{ijt} \leq \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$ or $\hat{\lambda}_{ijt} \geq \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$.*

Proof. Choose $(\hat{\lambda}, \hat{w}, \hat{z})$ to be an optimal solution with a minimal set $\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$. That is, there is no other optimal solution $(\hat{\lambda}', \hat{w}', \hat{z}')$ which has

$$\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}', \hat{w}', \hat{z}') \subsetneq \bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z}).$$

For resource i, we let $\hat{r}_i = \max\{r \,|\, r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})\}$, so that for all $r > \hat{r}_i$, $\xi_{it}(r) > 0$. Now suppose there exists a product j such that for $i \in j$ we have $\hat{\lambda}_{ijt} < \psi_{i,t+1}(\hat{r}_i)$, while for $l \in j$, we have $\hat{\lambda}_{ljt} > \psi_{l,t+1}(\hat{r}_l)$. In this case, we construct an optimal solution $(\bar{\lambda}, \bar{w}, \bar{z})$ with $\bigcup_{i \in \mathcal{I}} B_i(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq \bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$, which contradicts $(\hat{\lambda}, \hat{w}, \hat{z})$ being an optimal solution with a minimal set of binding constraints of type (14) amongst all optimal solutions.

Recall (from Lemma 1) that $\bar{v}_{i't}(r_{i'}) - \bar{v}_{i't}(r_{i'} - 1) \geq \bar{v}_{i't}(r_{i'} + 1) - \bar{v}_{i't}(r_{i'})$ for all i', $r_{i'} \in \mathcal{R}_{i'}$. Thus, we have

$$\psi_{i',t+1}(r_{i'}) \geq \psi_{i',t+1}(r_{i'} + 1) \geq 0, \quad \forall r_{i'} \in \mathcal{R}_{i'} \tag{19}$$

(≥ 0 as we set $\hat{v}_{i't}(r_{i'1} + 1) = \hat{v}_{i't}(r_{i'1})$; cf. Lemma 1). Also recall that we had assumed that $p_{jt} > 0$ without loss of generality.

Let $\epsilon = \min\left\{\psi_{i,t+1}(\hat{r}_i) - \hat{\lambda}_{ijt}, \hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l), \min\{\xi_{it}(r) \,|\, r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})\}\right\} > 0$, with the understanding that if $B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$ is empty, then $\min\{\xi_{it}(r) \,|\, r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})\} = \infty$.

Let $\bar{\lambda}$ be a vector which is equal to $\hat{\lambda}$ for all components except that $\bar{\lambda}_{ijt} = \hat{\lambda}_{ijt} + \delta$ and $\bar{\lambda}_{ljt} = \hat{\lambda}_{ljt} - \delta$, where $\delta \in (0, \epsilon)$. Note that $\bar{\lambda}$ satisfies constraint (16). Since $\bar{\lambda}_{ljt} > \hat{\lambda}_{ljt} - \epsilon \geq \psi_{l,t+1}(\hat{r}_l) \geq 0$, $\bar{\lambda}$ satisfies constraint (17).

Let $\bar{w}$ be equal to $\hat{w}$ for all resources and let $\bar{z}$ be equal to $\hat{z}$ for all resources except i and l. Therefore, for all $i' \notin \{i, l\}$, $(\bar{w}, \bar{z})$ satisfies constraints (14), (15) and (18), and $B_{i'}(\bar{\lambda}, \bar{w}, \bar{z}) = B_{i'}(\hat{\lambda}, \hat{w}, \hat{z})$.

For resources i and l, let $\bar{z}$ be equal to $\hat{z}$ for all products except j.

For resource i, let $\bar{z}_{ijtr} = p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+$ for $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$ and $\bar{z}_{ijtr} = \hat{z}_{ijtr}$ for $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$. We argue that $\bar{w}_{it}, \bar{z}_{ijtr}$ satisfy constraints (14), (15) and (18). By definition, $\bar{z}_{ijtr}$ satisfies constraints (15) and (18) for all $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$. For $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$, since $\hat{r}_i = \max\{r \,|\, r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})\}$, we have $\psi_{i,t+1}(r) \geq \psi_{i,t+1}(\hat{r}_i)$ (from 19). Therefore, $p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)] \leq p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i) + \delta] < p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i) + \epsilon] \leq 0 \leq \hat{z}_{ijtr} = \bar{z}_{ijtr}$. Consequently, $\bar{z}_{ijtr}$ satisfies constraints (15) and (18) for all $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$.

For $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, we have $\bar{z}_{ijtr} = p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(r) + \delta]^+ \leq p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+ + p_{jt}\delta < \hat{z}_{ijtr} + \epsilon \leq \hat{z}_{ijtr} + \xi_{it}(r)$. Therefore, for all $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, $\hat{w}_{it} \geq \sum_{k \ni i, k \neq j} \hat{z}_{iktr} + \hat{z}_{ijtr} + \Delta_{it}(r) + \xi_{it}(r) > \sum_{k \ni i} \bar{z}_{iktr} + \Delta_{it}(r)$, where we use the fact that $\bar{z}_{iktr} = \hat{z}_{iktr}$ for all $k \neq j$. So, as we set $\bar{w} = \hat{w}$ for all resources, $(\bar{w}, \bar{z})$ satisfies constraint (14) for all $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$ as a strict inequality.

On the other hand, for $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$ since we have $\bar{z}_{ijtr} = \hat{z}_{ijtr}$ and $\bar{w}_{it} = \hat{w}_{it}$, $(\bar{w}, \bar{z})$ satisfies constraint (14) as an equality. Therefore, $B_i(\bar{\lambda}, \bar{w}, \bar{z}) = B_i(\hat{\lambda}, \hat{w}, \hat{z})$. To summarize, $(\bar{w}, \bar{z})$ satisfies constraints (14), (15) and (18) for resource i and $B_i(\bar{\lambda}, \bar{w}, \bar{z}) = B_i(\hat{\lambda}, \hat{w}, \hat{z})$.

For resource l, we set $\bar{z}_{ljtr} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+$ for all $r \in \mathcal{R}_l$. Since $\bar{\lambda}_{ljt} < \hat{\lambda}_{ljt}$, we have $\hat{z}_{ljtr} \geq p_{jt}[\hat{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+ \geq p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+ = \bar{z}_{ljtr}$ for all $r \in \mathcal{R}_l$. On the other hand, since $\bar{z}_{lktr} = \hat{z}_{lktr}$ for all $k \neq j$ and $r \in \mathcal{R}_l$, and $\bar{w}_{lt} = \hat{w}_{lt}$, we have $\bar{w}_{lt} - [\sum_{k \in l} \bar{z}_{lktr} + \Delta_{lt}(r)] \geq \hat{w}_{lt} - [\sum_{k \in l} \hat{z}_{lktr} + \Delta_{lt}(r)] \geq 0$ for all $r \in \mathcal{R}_l$. Therefore, $(\bar{w}, \bar{z})$ satisfies constraints (14), (15) and (18) for resource l.

Since $\bar{w}_{lt} - [\sum_{k \in l} \bar{z}_{lktr} + \Delta_{lt}(r)] \geq \hat{w}_{lt} - [\sum_{k \in l} \hat{z}_{lktr} + \Delta_{lt}(r)]$ for all $r \in \mathcal{R}_l$, $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subseteq B_l(\hat{\lambda}, \hat{w}, \hat{z})$. Moreover, $\bar{z}_{l,j,t,\hat{r}_l} < \hat{z}_{l,j,t,\hat{r}_l}$ because $\bar{z}_{l,j,t,\hat{r}_l} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{i,t+1}(\hat{r}_l)]^+ = p_{jt}[\bar{\lambda}_{ljt} - \psi_{i,t+1}(\hat{r}_l)] < p_{jt}[\hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)] \leq p_{jt}[\hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)]^+ \leq \hat{z}_{l,j,t,\hat{r}_l}$. Therefore, we have $\bar{w}_{lt} - [\sum_{k \in l} \bar{z}_{l,k,t,\hat{r}_l} + \Delta_{lt}(\hat{r}_l)] > \hat{w}_{lt} - [\sum_{k \in l} \hat{z}_{l,k,t,\hat{r}_l} + \Delta_{lt}(\hat{r}_l)] = 0$, where the equality holds since $\hat{r}_l \in B_l(\hat{\lambda}, \hat{w}, \hat{z})$. Consequently, $\hat{r}_l \notin B_l(\bar{\lambda}, \bar{w}, \bar{z})$ and we have $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq B_l(\hat{\lambda}, \hat{w}, \hat{z})$. To summarize, $(\bar{w}, \bar{z})$ satisfies constraints (14), (15) and (18) for resource l and $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq B_l(\hat{\lambda}, \hat{w}, \hat{z})$.

Putting everything together, $(\bar{\lambda}, \bar{w}, \bar{z})$ is an optimal solution with

$$\bigcup_{i \in \mathcal{I}} B_i(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq \bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$$

and we get a contradiction. □

The following proposition shows that the upper bound on $\Phi_t(\bar{v})$ is tight.

Proposition 4. $\Phi_t(\bar{v}) = \Pi_t(\bar{v})$.

Proof. Lemma 3 implies that $\Phi_t(\bar{v}) \leq \Pi_t(\bar{v})$. On the other hand, letting $(\hat{\lambda}, \hat{w}, \hat{z})$ be as in Proposition 3, with minimal set $\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$ we have

$$\Pi_t(\bar{v}) = \sum_i \hat{w}_{it} = \sum_i \sum_{j \ni i} \hat{z}_{i,j,t,\hat{r}_i} + \Delta_{it}(\hat{r}_i) = \sum_j \sum_{i \in j} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)]^+ + \sum_i \Delta_{it}(\hat{r}_i),$$

where the second equality holds since $\hat{r}_i \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$ for all i, and the last equality holds since if $\hat{z}_{i,j,t,\hat{r}_i} > p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)]^+$, then we can decrease $\hat{z}_{i,j,t,\hat{r}_i}$ by a small positive number contradicting either the optimality of $(\hat{\lambda}, \hat{w}, \hat{z})$ or the fact that $(\hat{\lambda}, \hat{w}, \hat{z})$ is an optimal solution with a minimal set $\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$ amongst all optimal solutions. Let $\mathcal{J}_1 = \{j \mid \hat{\lambda}_{ijt} \geq \psi_{i,t+1}(\hat{r}_i) \, \forall i \in j\}$ and $\mathcal{J}_2 = \mathcal{J} \backslash \mathcal{J}_1$ where $\mathcal{J} = \{1, \ldots, n\}$. By Proposition 3, every product $j \in \mathcal{J}_2$ satisfies $\hat{\lambda}_{ijt} \le \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$. Therefore,

$$\begin{aligned}\Pi_t(\bar{v}) &= \sum_{j\in\mathcal{J}_1}\sum_{i\in j} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i)\\ &= \sum_j \sum_{i\in j} p_{jt}\hat{u}_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i)\\ &= \sum_j p_{jt}\hat{u}_{jt}[f_j - \sum_{i\in j}\psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i)\\ &\le \Phi_t(\bar{v})\end{aligned}$$

where we define $\hat{u}_{jt} = 1$ for $j \in \mathcal{J}_1$ and $\hat{u}_{jt} = 0$ for $j \in \mathcal{J}_2$. Note that the last equality follows from constraint (16). The last inequality holds since $\vec{r} = [\hat{r}_i], \vec{u} = [\hat{u}_{jt}]$ is feasible to problem (7): we trivially have $u_{jt} \le \hat{r}_i$ for all $j \in \mathcal{J}_2$ and $i \in j$. On the other hand, since $\hat{\lambda}$ is finite and $\psi_{i,t+1}(0) = \infty$, we have $r_i \ge 1$ for all $j \in \mathcal{J}_1$ and $i \in j$, and it follows that $u_{jt} \le \hat{r}_i$ for all $j \in \mathcal{J}_1$ and $i \in j$.

□

5.2 Proof of Proposition 2

By equation (7), constraint (2) can be written as $0 \ge \Phi_t(v)$ for all t, while Lemma 2 implies that there exists an optimal solution to (PL) such that $\Phi_t(v) = 0$ for all t. Therefore, we can write (PL) as

$$\begin{aligned}V^{PL} &= \min_{\{v|\Phi_t(v)=0\forall t\}} \sum_t \Phi_t(v) + \sum_i v_{i1}(r_{i1})\\ &= \min_{\{v|\Phi_t(v)=0\forall t\},\{\lambda\,|\,\sum_{i\in j}\lambda_{ijt}=f_j\,\forall t,j;\lambda_{ijt}\ge 0\,\forall t,j,i\in j\}} \sum_t\sum_i \Pi^{\lambda}_{it}(v) + \sum_i v_{i1}(r_{i1})\end{aligned}$$

where the last equality follows from Proposition 4 and (9). Now we drop the constraint $\Phi_t(v) = 0$, and as it is a minimization problem we get a lower bound on V^{PL}, that we call V. Using (8) and the fact that $\Pi^{\lambda}_{it}(v)$ appears in the objective function of a minimization problem, we have

$$V = \min_{\lambda,\pi,v} \sum_t\sum_i \pi_{it} + \sum_i v_{i1}(r_{i1})$$

s.t.

$$\pi_{it} \ge \sum_{j\ni i} p_{jt}u_{ijt}[\lambda_{ijt} + v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] + v_{i,t+1}(r_i) - v_{it}(r_i)$$

$$\forall t, i, r_i \in \mathcal{R}_i, \vec{u}_{it} \in \mathcal{U}_i(r_i)$$

$$\sum_{i\in j}\lambda_{ijt} = f_j \quad \forall t, j$$

$$\lambda_{ijt} \ge 0, v_{it}(\cdot) \ge 0, v_{i,\tau+1}(\cdot) = 0.$$

Letting $\pi_{it} = \theta_{it} - \theta_{i,t+1}$ with $\theta_{i,\tau+1} = 0$, the above objective function becomes $\sum_i \theta_{i1} + v_{i1}(r_{i1})$ while the constraints become $\theta_{it} + v_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_{ijt}[\lambda_{ijt} + v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] + \theta_{i,t+1} + v_{i,t+1}(r_i)$. Finally, letting $\nu_{it}(r_i) = \theta_{it} + v_{it}(r_i)$, we have

$$
\begin{aligned}
V \quad & = \min_{\lambda,\nu} \sum_i \nu_{i1}(r_{i1}) \\
\text{s.t.} \quad & \\
& \nu_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_{ijt}[\lambda_{ijt} + \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)] + \nu_{i,t+1}(r_i) \\
& \qquad\qquad\qquad\qquad \forall t, i, r_i \in \mathcal{R}_i, \vec{u}_{it} \in \mathcal{U}_i(r_i) \\
& \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall t, j \\
& \lambda_{ijt} \geq 0, \nu_{it}(\cdot) \geq 0, \nu_{i,\tau+1}(\cdot) = 0
\end{aligned}
$$

where $\nu_{i,t}(\cdot) \geq 0$ follows from the fact that $\nu_{i,\tau+1}(\cdot) = 0$ and $\vec{u}_{it} = \emptyset$ is one of the controls. This is exactly (LR), the linear programming formulation of the Lagrangian relaxation. So $V^{LR} = V \leq V^{PL}$.

But $V^{LR} \geq V^{PL}$ also as the following simple argument shows: Consider a feasible solution λ, ν to (LR). For a given $t, \vec{r} = [r_i]$ and $\vec{u}_t \in \mathcal{U}(\vec{r})$, note that $\vec{u}_t \in \mathcal{U}_i(r_i)$. Summing up constraints (5) for r_i and $\vec{u}_t \in \mathcal{U}_i(r_i)$ for all i,

$$
\begin{aligned}
\sum_i \nu_{it}(r_i) \quad & \geq \quad \sum_i \sum_{j \ni i} p_{jt} u_{jt}[\lambda_{ijt} + \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)] + \sum_i \nu_{i,t+1}(r_i) \\
& = \quad \sum_j p_{jt} u_{jt}[f_j + \sum_{i \in j} \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)] + \sum_i \nu_{i,t+1}(r_i)
\end{aligned}
$$

where the equality holds since $\sum_{i \in j} \lambda_{ijt} = f_j$. So ν is a feasible solution to (PL) with the same value.

Therefore $V^{LR} = V^{PL}$. $\square$

5.3 Polynomial-time separation for (PL)

The separation for (PL) can be done by solving the compact linear program (S) for a given set of $\bar{v}$ variables. If its optimal objective function value $\Pi_t(\bar{v}) \leq 0$ for all t then $\bar{v}$ is feasible in (PL). If $\Pi_t(\bar{v}) > 0$ for some t, then we find a state-action pair $(\vec{r} \in \mathcal{R}, \vec{u}_t \in \mathcal{U}(\vec{r}))$ that violates constraint (2) in the following manner.

Separation Algorithm

Step 1: Let $(\hat{\lambda}^0, \hat{w}^0, \hat{z}^0)$ be an optimal solution to (S). Set $k = 0$.

Step 2: Let $\{\hat{r}_i^k \,|\, \forall i\}$ be as defined in Proposition 3.
If, for all j, $\hat{\lambda}_{ijt}^k \leq \psi_{i,t+1}(\hat{r}_i^k)$ for all $i \in j$ or $\hat{\lambda}_{ijt}^k \geq \psi_{i,t+1}(\hat{r}_i^k)$ for all $i \in j$, set $u_{jt} = 1$ for all $j \in \mathcal{J}_1$ and $u_{jt} = 0$ for all $j \in \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are as defined in Proposition 4. Set $\vec{r} = \{\hat{r}_i^k \,|\, \forall i\}$ and $\vec{u}_t = \{u_{jt} \,|\, \forall j\}$ and stop.
Else, pick a product j such that for $i \in j$, we have $\hat{\lambda}_{ijt}^k < \psi_{i,t+1}(\hat{r}_i)$, while for $l \in j$, we have $\hat{\lambda}_{ljt}^k > \psi_{l,t+1}(\hat{r}_l^k)$. Let $(\bar{\lambda}^k, \bar{w}^k, \bar{z}^k)$ be as described in Proposition 3.

Step 3: Set $\hat{\lambda}^{k+1} = \bar{\lambda}^k, \hat{w}^{k+1} = \bar{w}^k$ and $\hat{z}^{k+1} = \bar{z}^k$. Set $k = k + 1$ and go to Step 2.

By Proposition 3, $(\hat{\lambda}^k, \hat{w}^k, \hat{z}^k)$ is an optimal solution to (S) for all k. Proposition 3 also implies that $(\hat{\lambda}^{k+1}, \hat{w}^{k+1}, \hat{z}^{k+1})$ has strictly fewer number of binding constraints of type (14) than $(\hat{\lambda}^k, \hat{w}^k, \hat{z}^k)$. Since the number of binding constraints of type (14) in any optimal solution is at least m and at most $\sum_i r_{i1}$, Separation Algorithm terminates in polynomial time. Finally, by Proposition 4, $\vec{u}_t \in \mathcal{U}(\vec{r})$.

6 Conclusions

We make the following research contributions in this paper: (1) We show that the approximate dynamic programming approach with piecewise-linear basis functions (Farias and van Roy [2]) and the Lagrangian relaxation approach (Topaloglu [9]) are in fact equivalent. This result shows that there might be surprising connections between the approximate dynamic programming approach and Lagrangian relaxation for complicated dynamic programs, and one can benefit from unifying forces as it were. (2) We show that separation problem for the piecewise-linear approximation is solvable in polynomial-time. (3) We show that there exists a separable concave approximation that yields the tightest upper bound among all separable piecewise-linear approximations to the value function. This implies that the Lagrangian relaxation approach obtains the tightest upper bound among all separable piecewise-linear approximations that are upper bounds on the value function.

As to computational impact, we solved the piecewise-linear approximation using the linear-programming based separation we describe in this paper, but our results suggest that the subgradient procedure for the Lagrangian relaxation is still faster. This is in line with the computational studies reported in Farias and van Roy [2], and not too surprising as solvability by separation is based on the ellipsoid algorithm that is well known to be slow in practice. Tong and Topaloglu [8] make a similar observation for the affine relaxation of Adelman [1]; they also find that the Lagrangian relaxation approach described in Kunnumkal and Topaloglu [4] is more efficient. Improving the efficiency of the separation, say by a faster, combinatorial, algorithm, would be an interesting area for future research.

Our proof technique uses the following characteristics of the resource-level RM dynamic programs: decreasing marginal values and a threshold-value type of controls. It would be interesting to see if any other problem exhibits this close connection between approximate dynamic programming and Lagrangian relaxation.

References

[1] Adelman, D. 2007. Dynamic bid-prices in revenue management. *Operations Research* **55**(4) 647–661.

[2] Farias, V. F., B. van Roy. 2007. An approximate dynamic programming approach to network revenue management. Tech. rep., Sloan School of Management, MIT, Massachussetts, MA.

[3] Grötschel, M., L. Lovász, A. Schrijver. 1988. *Geometric Algorithms and Combinatorial Optimization*, vol. 2. Springer.

[4] Kunnumkal, S., H. Topaloglu. 2010. Computing time-dependent bid prices in network revenue management problems. *Transportation Science* **44** 38–62.

[5] Meissner, J., A. K. Strauss. 2012. Network revenue management with inventory-sensitive bid prices and customer choice. *European Journal of Operational Research* **216**(2) 459–468.

[6] Talluri, K. T. 2008. On bounds for network revenue management. Tech. Rep. WP-1066, UPF.

[7] Talluri, K. T., G. J. van Ryzin. 2004. *The Theory and Practice of Revenue Management*. Kluwer, New York, NY.

[8] Tong, C., H. Topaloglu. 2011. On approximate linear programming approach for network revenue management problems. Tech. rep., ORIE, Cornell University.

[9] Topaloglu, H. 2009. Using Lagrangian relaxation to compute capacity-dependent bid prices in network revenue management. *Operations Research* **57** 637–649.

[10] Zhang, D., D. Adelman. 2009. An approximate dynamic programming approach to network revenue management with customer choice. *Transportation Science* **43**(3) 381–394.

Appendix

Proof of Lemma 1:

Our analysis is essentially an adaptation of analogous structural results for the revenue management problem on a single resource (Talluri and van Ryzin [7]). We introduce some notation to simplify the expressions. Fixing a resource l, we let $\mathcal{R}_l(r_l) = \{\vec{x} \in \mathcal{R} \,|\, x_l = r_l\}$ be the set of capacity vectors where the capacity on resource l is fixed at r_l. Given a separable piecewise-linear approximation $v = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$, we let

$$\epsilon_{lt}(r_l, v) \quad = \min_{\vec{r} \in \mathcal{R}_l(r_l), \vec{u}_t \in \mathcal{U}(\vec{r})} \Big\{ \sum_i v_{it}(r_i) - \sum_j p_{jt} u_{jt} \big[f_j + \sum_{i \in j} [v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] \big] - \sum_i v_{i,t+1}(r_i) \Big\}$$

where the argument v emphasizes the dependence on the given approximation. Note that if $v = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ is feasible to (PL), then $\epsilon_{it}(r_i, v) \geq 0$ for all t, i and $r_i \in \mathcal{R}_i$. We begin with a preliminary result.

Lemma 5. *There exists an optimal solution $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ to (PL) such that for all t, i and $r_i \in \mathcal{R}_i$, we have $\epsilon_{it}(r_i, \hat{v}) = 0$.*

Proof. Let $v = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ be an optimal solution to problem (PL). Let s be the largest time index such that there exists a resource l and $r_l \in \mathcal{R}_l$ with $\epsilon_{ls}(r_l, v) > 0$. Since v is feasible, this means that $\epsilon_{it}(r_i, v) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. We consider decreasing $v_{ls}(r_l)$ alone by $\epsilon_{ls}(r_l, v)$ leaving all the other components of v unchanged. That is, let $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ where

$$\hat{v}_{it}(x) = \begin{cases} v_{it}(x) - \epsilon_{it}(x, v) & \text{if } i = l, t = s, x = r_l \\ v_{it}(x) & \text{otherwise.} \end{cases} \tag{20}$$

Note that since $\hat{v}_{it}(r_i) \leq v_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$, we have $\sum_i \hat{v}_{i1}(r_{i1}) \leq \sum_i v_{i1}(r_{i1})$. Next, we show that $\hat{v}$ is feasible. Since $\hat{v}$ differs from v only in one component, we only have to check those constraints where $\hat{v}_{ls}(r_l)$ appears. Observe that $\hat{v}_{ls}(r_l)$ appears only in the constraints for time periods $s-1$ and s. For time period $s-1$, we have

$$\begin{aligned} &\sum_j p_{j,s-1} u_{j,s-1} \big[f_j + \sum_{i \in j} \hat{v}_{is}(r_i - 1) \big] + \sum_i [1 - \sum_{j \ni i} p_{j,s-1} u_{j,s-1}] \hat{v}_{is}(r_i) \\ &\quad \leq \sum_j p_{j,s-1} u_{j,s-1} \big[f_j + \sum_{i \in j} v_{is}(r_i - 1) \big] + \sum_i [1 - \sum_{j \ni i} p_{j,s-1} u_{j,s-1}] v_{is}(r_i) \\ &\quad \leq \sum_i v_{i,s-1}(r_i) \\ &\quad = \sum_i \hat{v}_{i,s-1}(r_i) \end{aligned}$$

for all $\vec{r} \in \mathcal{R}$ and $\vec{u}_{s-1} \in \mathcal{U}(\vec{r})$, where the first inequality follows since $\hat{v}_{is}(r_i) \le v_{is}(r_i)$ and $\sum_{j \ni i} p_{j,s-1} u_{j,s-1} \le 1$, the second inequality follows from the feasibility of v and the equality follows from (20). For time period s, $\hat{v}_{ls}(r_l)$ appears only in constraints corresponding to $\vec{r} \in \mathcal{R}_l(r_l)$. For $\vec{r} \in \mathcal{R}_l(r_l)$, we have

$$\begin{aligned}
&\sum_i \hat{v}_{is}(r_i) \\
&\quad = \sum_i v_{is}(r_i) - \epsilon_{ls}(r_l, v) \\
&\quad \ge \sum_j p_{js} u_{js} \big[f_j + \sum_{i \in j} v_{i,s+1}(r_i - 1) - v_{i,s+1}(r_i) \big] + \sum_i v_{i,s+1}(r_i) \\
&\quad = \sum_j p_{js} u_{js} \big[f_j + \sum_{i \in j} \hat{v}_{i,s+1}(r_i - 1) - \hat{v}_{i,s+1}(r_i) \big] + \sum_i \hat{v}_{i,s+1}(r_i)
\end{aligned}$$

for all $\vec{u}_s \in \mathcal{U}(\vec{r})$, where the inequality follows from the definition of $\epsilon_{ls}(r_l, v)$ and the last equality follows from (20). Therefore $\hat{v}$ is feasible, which implies that $\epsilon_{it}(r_i, \hat{v}) \ge 0$ for all t, i and $r_i \in \mathcal{R}_i$. Next, we note from (20) that $\epsilon_{it}(r_i, \hat{v}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. For time period s, since $\hat{v}_{is}(r_i) \le v_{is}(r_i)$ and $\hat{v}_{i,s+1}(r_i) = v_{i,s+1}(r_i)$, it follows that $\epsilon_{is}(r_i, \hat{v}) \le \epsilon_{is}(r_i, v)$. Therefore, if $\epsilon_{is}(r_i, v)$ was zero, then $\epsilon_{is}(r_i, \hat{v})$ is also zero. Moreover, $\epsilon_{ls}(r_l, \hat{v}) = 0 < \epsilon_{ls}(r_l, v)$.

To summarize, $\hat{v}$ is an optimal solution with $\epsilon_{it}(r_i, \hat{v}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$ and $|\{\epsilon_{is}(r_i, \hat{v}) \,|\, \epsilon_{is}(r_i, \hat{v}) > 0\}| < |\{\epsilon_{is}(r_i, v) \,|\, \epsilon_{is}(r_i, v) > 0\}|$. We repeat the above procedure finitely many times to obtain an optimal solution $\hat{v}$ with $\epsilon_{it}(r_i, \hat{v}) = 0$ for all $t \ge s$, i and $r_i \in \mathcal{R}_i$. Repeating the entire procedure for time periods $s-1, \ldots, 1$ completes the proof. □

We are ready to prove Lemma 1. By Lemma 5, we can pick an optimal solution $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ such that $\epsilon_{it}(r_i, \hat{v}) = 0$ for all t, i and $r_i \in \mathcal{R}_i$. The proof proceeds by induction on the time periods. It is easy to see that the result holds for time period τ. Fix a resource l and assume that statements (i) and (ii) of the lemma hold for all time periods $s > t$. We show below that statements (i) and (ii) hold for time period t as well.

Since $\hat{v}_{it}(-1) = -\infty$, statement (i) holds trivially for $r_l = 0$. For $r_l = 1$, Lemma 5 implies that there exists $\vec{x} \in \mathcal{R}_l(0)$ and $\vec{u}_t \in \mathcal{U}(\vec{x})$ such that

$$\begin{aligned}
\hat{v}_{lt}(0) + \sum_{i \ne l} \hat{v}_{it}(x_i) = &\sum_{j \not\ni l} p_{jt} u_{jt} [f_j + \sum_{i \ne l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)]] \\
&+ \hat{v}_{l,t+1}(0) + \sum_{i \ne l} \hat{v}_{i,t+1}(x_i). \qquad (21)
\end{aligned}$$

where $\mathbb{1}_{[\cdot]}$ denotes the indicator function and we use the fact that since $x_l = 0$, $u_{jt} = 0$ for all $j \ni l$. Next, consider the capacity vector $\vec{y}$ with $y_i = x_i$ for $i \ne l$ and $y_l = r_l = 1$. Since

$\vec{x} \le \vec{y}, \mathcal{U}(\vec{x}) \subset \mathcal{U}(\vec{y})$ and it follows that $\vec{u}_t \in \mathcal{U}(\vec{y})$. Since $\hat{v}$ is feasible, we have

$$\hat{v}_{lt}(1) + \sum_{i \neq l} \hat{v}_{it}(x_i) \ge \sum_{j \not\ni l} p_{jt} u_{jt} [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)]] \\ + \hat{v}_{l,t+1}(1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \tag{22}$$

Subtracting (21) from (22), we have $\hat{v}_{lt}(1) - \hat{v}_{lt}(0) \ge \hat{v}_{l,t+1}(1) - \hat{v}_{l,t+1}(0)$.

We next show that statement (i) holds for $r_l \in \mathcal{R}_l \backslash \{0, 1\}$. By Lemma 5, there exists $\vec{x} \in \mathcal{R}_l(r_l - 1)$ and $\vec{u}_t \in \mathcal{U}(\vec{x})$ such that

$$\begin{aligned} &\hat{v}_{lt}(r_l - 1) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\ &\quad = \sum_j p_{jt} u_{jt} [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l - 2) - \hat{v}_{l,t+1}(r_l - 1)]] \\ &\qquad + \hat{v}_{l,t+1}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \end{aligned} \tag{23}$$

Now, consider the capacity vector $\vec{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $\vec{x} \le \vec{y}$, $\mathcal{U}(\vec{x}) \subset \mathcal{U}(\vec{y})$ and it follows that $\vec{u}_t \in \mathcal{U}(\vec{y})$. Since $\hat{v}$ is feasible, we have

$$\begin{aligned} &\hat{v}_{lt}(r_l) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\ &\quad \ge \sum_j p_{jt} u_{jt} [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l)]] \\ &\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \end{aligned} \tag{24}$$

Subtracting (23) from (24), we get

$$\begin{aligned} &\hat{v}_{lt}(r_l) - \hat{v}_{lt}(r_l - 1) \\ &\quad \ge \sum_j p_{jt} u_{jt} \mathbb{1}_{[l \in j]} [2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2)] + \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1) \\ &\quad \ge \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1). \end{aligned} \tag{25}$$

Note that the last inequality follows, since by induction assumption (ii), we have

$$2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2) \ge 0.$$

Next, we show that statement (ii) holds for time period t. Since $\hat{v}_{it}(-1) = -\infty$, statement (ii) holds trivially for $r_l = 0$. For $r_l \in \mathcal{R}_l \backslash \{0, r_{l1}\}$, Lemma 5 implies that there exists

$\vec{x} \in \mathcal{R}_l(r_l+1)$ and $\vec{u}_t \in \mathcal{U}(\vec{x})$ such that

$$\begin{aligned}
&\hat{v}_{lt}(r_l+1) + \sum_{i\neq l} \hat{v}_{it}(x_i) \\
&\quad = \sum_j p_{jt}u_{jt}[f_j + \sum_{i\neq l} \mathbb{1}_{[i\in j]}[\hat{v}_{i,t+1}(x_i-1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l\in j]}[\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l+1)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l+1) + \sum_{i\neq l} \hat{v}_{i,t+1}(x_i). \qquad (26)
\end{aligned}$$

Now consider the capacity vector $\vec{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $r_l \geq 1$, $u_{jt} \leq r_l$ for all $j \ni l$. Since $y_i = x_i$ for $i \neq l$ and $\vec{u}_t \in \mathcal{U}(\vec{x})$, we have that $u_{jt} \leq y_i$ for all $j \ni i$. That is, we have $\vec{u}_t \in \mathcal{U}(\vec{y})$. Since $\hat{v}$ is feasible, we have

$$\begin{aligned}
&\hat{v}_{lt}(r_l) + \sum_{i\neq l} \hat{v}_{it}(x_i) \\
&\quad \geq \sum_j p_{jt}u_{jt}[f_j + \sum_{i\neq l} \mathbb{1}_{[i\in j]}[\hat{v}_{i,t+1}(x_i-1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l\in j]}[\hat{v}_{l,t+1}(r_l-1) - \hat{v}_{l,t+1}(r_l)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i\neq l} \hat{v}_{i,t+1}(x_i). \qquad (27)
\end{aligned}$$

Subtracting (27) from (26), we get

$$\begin{aligned}
&\hat{v}_{lt}(r_l+1) - \hat{v}_{lt}(r_l) \\
&\quad \leq \sum_j p_{jt}u_{jt}\mathbb{1}_{[l\in j]}[2\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l-1)] + \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l) \\
&\quad \leq 2\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l-1) + \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l) \\
&\quad = \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l-1) \\
&\quad \leq \hat{v}_{lt}(r_l) - \hat{v}_{lt}(r_l-1).
\end{aligned}$$

Note that the second inequality above follows, since by induction assumption (ii), $2\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l+1) - \hat{v}_{l,t+1}(r_l-1) \geq 0$ and $\sum_j p_{jt}u_{jt}\mathbb{1}_{[l\in j]} \leq 1$ and the last inequality follows from (25). Finally, for $r_l = r_{l1}$, following a similar analysis, we get

$$\begin{aligned}
\hat{v}_{lt}(r_{l1}) - \hat{v}_{lt}(r_{l1}-1) \geq &\sum_j p_{jt}u_{jt}\mathbb{1}_{[l\in j]}[2\hat{v}_{l,t+1}(r_{l1}-1) - \hat{v}_{l,t+1}(r_{l1}) - \hat{v}_{l,t+1}(r_{l1}-2)] \\
&+ \hat{v}_{l,t+1}(r_{l1}) - \hat{v}_{l,t+1}(r_{l1}-1).
\end{aligned}$$

By induction assumption (ii), $2\hat{v}_{l,t+1}(r_{l1}-1) - \hat{v}_{l,t+1}(r_{l1}) - \hat{v}_{l,t+1}(r_{l1}-2) \geq 0$ and $\hat{v}_{l,t+1}(r_{l1}) - \hat{v}_{l,t+1}(r_{l1}-1) \geq \hat{v}_{l,t+1}(r_{l1}+1) - \hat{v}_{l,t+1}(r_{l1})$. We have

$$\hat{v}_{lt}(r_{l1}) - \hat{v}_{lt}(r_{l1}-1) \geq \hat{v}_{l,t+1}(r_{l1}+1) - \hat{v}_{l,t+1}(r_{l1}) = 0 = \hat{v}_{lt}(r_{l1}+1) - \hat{v}_{lt}(r_{l1}).$$

Therefore, statements (i) and (ii) hold at time period t for resource l. This completes the proof since resource l was an arbitrary choice. □

Proof of Lemma 2:

Let $\hat{v} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ be as in Lemma 5. Fixing a resource l, we have

$$\begin{aligned}
&\max_{\vec{r}\in\mathcal{R},\vec{u}_t\in\mathcal{U}(\vec{r})} \left\{ \sum_j p_{jt}u_{jt}[f_j + \sum_{i\in j} \hat{v}_{i,t+1}(r_i-1) - \hat{v}_{i,t+1}(r_i)] + \sum_i \hat{v}_{i,t+1}(r_i) - \sum_i \hat{v}_{it}(r_i) \right\} \\
&= \max_{r_l\in\mathcal{R}_l} \left\{ \max_{\vec{r}\in\mathcal{R}_l(r_l),\vec{u}_t\in\mathcal{U}(\vec{r})} \left\{ \sum_j p_{jt}u_{jt}[f_j + \sum_{i\in j} \hat{v}_{i,t+1}(r_i-1) - \hat{v}_{i,t+1}(r_i)] \right.\right. \\
&\qquad\qquad \left.\left. + \sum_i \hat{v}_{i,t+1}(r_i) - \sum_i \hat{v}_{it}(r_i) \right\} \right\} \\
&= \max_{r_l\in\mathcal{R}_l} [-\epsilon_{lt}(r_l,\hat{v})] = 0,
\end{aligned}$$

where the last equality follows from Lemma 5. □

Proof of Proposition 1:

Suppose for product j and time t, we have $\sum_{i\in j}\lambda_{ijt} > f_j$. This implies that for some $i \in j$, $\lambda_{ijt} > 0$. Let $\delta = \min\{\lambda_{ijt}, \sum_{i\in j}\lambda_{ijt} - f_j\} > 0$ and let $\hat{\lambda}$ be a vector which is equal to λ for all components except that $\hat{\lambda}_{ijt} = \lambda_{ijt} - \delta < \lambda_{ijt}$.

Note that $\vartheta^{\hat{\lambda}}_{i1}(r_{i1}) \le \vartheta^{\lambda}_{i1}(r_{i1})$ as we have reduced the revenue associated with product j at time period t, keeping all other product revenues the same. As the first part of the right hand side of (3) is unaffected by this change, we get

$$V^{\hat{\lambda}}_1(\vec{r}_1) \le V^{\lambda}_1(\vec{r}_1).$$

If after performing this step, we still have $\sum_{i\in j}\hat{\lambda}_{ijt} > f_j$ for some product j and time t, we can repeat this step for another resource $i \in j$ until we have $\sum_{i\in j}\lambda_{ijt} \le f_j$ for all j and t.

Now suppose for some product j and time t, $\sum_{i\in j}\lambda_{ijt} < f_j$. Fix $i \in j$ and let $\hat{\lambda}$ be a vector which is equal to λ for all components, except that $\hat{\lambda}_{ijt} = \lambda_{ijt} + \delta$, where $\delta = f_j - \sum_{i\in j}\lambda_{ijt} > 0$.

We have

$$\vartheta^{\hat{\lambda}}_{i1}(r_{i1}) \le \vartheta^{\lambda}_{i1}(r_{i1}) + \delta.$$

This is because, by increasing the revenue associated with product j at time t by δ, while keeping all other product revenues the same, we cannot increase the optimal expected revenue from resource i by more than δ. However,

$$[f_j - \sum_{i\in j}\hat{\lambda}_{ijt}]^+ = [f_j - \sum_{i\in j}\lambda_{ijt}]^+ - \delta$$

and so

$$V^{\hat{\lambda}}_1(\vec{r}_1) \le V^{\lambda}_1(\vec{r}_1).$$

By repeating this step, if necessary, we obtain an optimal solution that satisfies $\sum_{i\in j}\lambda_{ijt} = f_j$ for all j and t.

Finally, we show that there exists optimal Lagrange multipliers $\lambda_{ijt} \geq 0$ for all t, j and $i \in j$. Suppose $\lambda_{ijt} < 0$. Since the Lagrange multipliers sum up to f_j, there exists $l \in j$ such that $\lambda_{ljt} > 0$. Let $\delta = \min\{[\lambda_{ijt}]^+, \lambda_{ljt}\} > 0$ and let $\hat{\lambda}$ be a vector which is equal to λ for all components except that $\hat{\lambda}_{ijt} = \lambda_{ijt} + \delta$ and $\hat{\lambda}_{ljt} = \lambda_{ljt} - \delta$. Observe that $\vartheta_{i1}^{\hat{\lambda}}(r_{i1}) \leq \vartheta_{i1}^{\lambda}(r_{i1})$ and $\vartheta_{l1}^{\hat{\lambda}}(r_{l1}) \leq \vartheta_{l1}^{\lambda}(r_{l1})$ so that $V_1^{\hat{\lambda}}(\vec{r}_1) \leq V_1^{\lambda}(\vec{r}_1)$. If there is still some Lagrange multiplier that is negative, we repeat the step until we have $\lambda_{ijt} \geq 0$ for all t, j and $i \in j$. $\square$

Proof of Lemma 4:

Note that an optimal solution to problem (8) satisfies $u_{ijt} = \mathbb{1}_{[\lambda_{ijt} \geq \psi_{i,t+1}(r_i)]}$ for all $j \ni i$, where we use the fact that $\psi_{i,t+1}(0) = \infty$. Therefore, using the convention that $0 \times -\infty = 0$, problem (8) can be written as $\max_{r\in\mathcal{R}_i}\sum_{j\ni i} p_{jt}\mathbb{1}_{[\lambda_{ijt}\geq\psi_{i,t+1}(r)]}[\lambda_{ijt} - \psi_{i,t+1}(r)] + \Delta_{it}(r)$. On the other hand, there exists an optimal solution $\big(\hat{w}_{it}, \{\hat{z}_{ijtr} \,|\, \forall j \ni i, r \in \mathcal{R}_i\}\big)$ to (S_i) such that $\hat{z}_{ijtr} = p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)]^+ = p_{jt}\mathbb{1}_{[\lambda_{ijt}\geq\psi_{i,t+1}(r)]}[\lambda_{ijt} - \psi_{i,t+1}(r)]$ for all $j \ni i$ and $r \in \mathcal{R}_i$. Moreover, there exists an $\hat{r} \in \mathcal{R}_i$ such that $\hat{w}_{it} = \sum_{j\ni i}\hat{z}_{ijt\hat{r}} + \Delta_{it}(\hat{r})$. Therefore, $\hat{w}_{it} = \max_{r\in\mathcal{R}_i}\sum_{j\ni i}\hat{z}_{ijtr} + \Delta_{it}(r)$. $\square$