A tractable consideration set structure for network revenue management

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February 15, 2012

Abstract

The dynamic program for choice network RM is intractable and approximated by a deterministic linear program called the $CDLP$. When the segment consideration sets overlap, the $CDLP$ is difficult to solve. A weaker formulation ($SDCP+$) is tractable and approximates the $CDLP$ value very closely. We show that if the segment consideration sets follow a tree structure, the two problems are equivalent, and give a counterexample to show that cycles can induce a gap between $CDLP$ and the relaxation.

Keywords: discrete-choice models, network revenue management, consideration sets

1 Introduction and literature review

Revenue management (RM) is the control of the sale of a limited quantity of a resource (hotel rooms for a night, airline seats, advertising slots etc.) to a heterogenous population with different valuations for a unit of the resource. The resource is perishable, and for simplicity's sake, we assume that it perishes at a fixed point of time in the future. Customers are independent of each other and arrive randomly during a sale period, and demand one unit of resource each. Sale is online, and the firm has to decide which products at what price it should offer, the tradeoff being selling too much at too low a price early and running out of capacity, or, losing too many price-sensitive customers and ending up with excess unsold inventory.

In industries such as hotels, airlines and media, the products consume bundles of different resources (multi-night stays, multi-leg itineraries) and the decision on whether to offer a particular product at a certain price depends on the expected future demand and current inventory levels for all the resources used by the product, and hence indirectly, all the resources in the network. Network revenue management (network RM) is control based on the demands for the entire network. Chapter 3 of Talluri and van Ryzin [14] contains all the necessary background on network RM.

RM incorporating more realistic models of customer behavior, as customers choosing from an offer set, have recently become popular (Talluri and van Ryzin [13], Gallego, Iyengar, Phillips, and Dubey [3], Liu and van Ryzin [7], Kunnumkal and Topaloglu [6], Zhang and Adelman [15], Meissner and Strauss [8], Bodea, Ferguson, and Garrow [1], Bront, Méndez-Díaz, and Vulcano [2], Méndez-Díaz, Bront, Vulcano, and Zabala [10], Kunnumkal [5]).

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Gallego et al. [3] and Liu and van Ryzin [7] propose a linear program called Choice Deterministic Linear Program ($CDLP$), while Talluri [12] proposed a formulation called Segment-based Deterministic Concave Program ($SDCP$) that is weaker than the upper bound resulting from the $CDLP$ but coincides for non-overlapping segment consideration sets. Gallego, Ratliff, and Shebalov [4] taking a similar approach but specialized to the multinomial logit (MNL) model of choice propose a compact linear program called the Sales Based Linear Program ($SBLP$) that also coincides with $CDLP$ for non-overlapping segment consideration sets.

The $CDLP$ formulation has an exponential number of columns, exponential in the number of products, and the solution strategy is to use column generation; but finding an entering column is NP-hard even in restrictive cases such as the MNL model of choice with just two segments and overlapping consideration sets (Bront et al. [2], Rusmevichientong, Shmoys, and Topaloglu [11]).

$SDCP$ is tractable when consideration sets are small, but its performance is poor when segment consideration sets overlap (i.e., the bound is significantly looser than $CDLP$). Meissner, Strauss, and Talluri [9] extend the $SDCP$ formulation, that we call $SDCP+$ here, to obtain progressively tighter relaxations of $CDLP$ for the case of overlapping consideration sets by adding *product cuts* that interpret the linear programming decision variables as randomization rules. These constraints are easy to generate and work for general discrete-choice models. Talluri [12] specializes the $SBLP$ formulation of Gallego et al. [4] to obtain a more compact formulation (that we call $SBLP+$) that is exponential only in the size of the intersections of the consideration sets. In numerical testing $SDCP+$ and $SBLP+$ achieve the $CDLP$ value in many instances despite being an order of magnitude more compact.

This paper is an attempt to understand when the more compact formulations $SDCP+$ and $SBLP+$ achieve the intractable $CDLP$ formulation value, identifying the structure of the overlapping consideration sets that make the $SDCP+$ and $CDLP$ formulations exactly equal. We show that if the segment consideration sets follow a certain tree structure, the two problems are equivalent, and give a counterexample to show that cycles can induce a gap between $CDLP$ and the $SDCP+$ relaxation.

The remainder of the paper is organized as follows: In §2 we introduce the notation, the demand model and the basic dynamic program. In §3 we state two approximations of the dynamic program, namely the $CDLP$ and the $SDCP$ with product constraints, followed by the presentation of the main structural results that we prove in this paper. §4 contains our numerical results using the new methods, and we present our conclusions in §5.

2 Model and notation

A product is a specification of a price and a combination of resources to be consumed. For example, a product could be an itinerary-fare class combination for an airline network, where an itinerary is a combination of flight legs; in a hotel network, a product would be a multi-night stay for a particular room type at a certain price point. Time is discrete and assumed to consist of T intervals, indexed by t. We assume that the booking horizon begins at time 0 and that all the resources perish instantaneously at time T. We make the standard assumption that the time intervals are fine enough so that the probability of more than one customer arriving in any single time period is negligible. The underlying network has m resources (indexed by i) and n products (indexed by j), and we refer to the set of all resources as I and the set of all products as J. A product j uses a subset of resources, and is identified (possibly) with a set of sale restrictions or features and a revenue of r_j. A resource i is said to be in product j ($i \in j$) if j uses resource i. The resources used by j are represented by $a_{ij} = 1$ if $i \in j$, and $a_{ij} = 0$ if $i \notin j$, or alternately with the 0-1 incidence vector $\vec{A}_j$ of product j. Let A denote the resource-product incidence matrix; columns of A are then $\vec{A}_j$. We denote the vector of capacities at time t as $\vec{c}_t$, so the initial set of capacities at time 0 is $\vec{c}_0$. The vector $\vec{1}$ is a vector of all ones, and $\vec{0}$ is a vector of all zeroes (dimension appropriate to the context).

We assume there are $\mathcal{L} := \{1, \ldots, L\}$ customer segments, each with distinct purchase behavior. In each period, there is a customer arrival with probability λ. A customer belongs to segment l with probability p_l. We denote $\lambda_l = p_l \lambda$ and assume $\sum_l p_l = 1$, so $\lambda = \sum_l \lambda_l$. We are assuming time-homogenous arrivals (homogenous in rates and segment mix), but the model and all solution methods in this paper can be transparently extended to the case when rates and mix change by period. Each segment l has a *consideration set* $C_l \subseteq J$ of products that it considers for purchase. We assume this consideration set is known to the firm (by a previous process of estimation and analysis), and the consideration sets for different segments can overlap.

In each period the firm offers a subset S of its products for sale, called the *offer set*. Given an offer set S, an arriving customer purchases a product j in the set S or decides not to purchase. The no-purchase option is indexed by 0 and is always present for the customer.

A segment-l customer is indifferent to a product outside his consideration set; i.e., his choice probabilities are not affected by the availability of products $j \in J \setminus C_l$. A segment-l customer purchases $j \in S$ with given probability $P^l_j(S)$. This is a set-function defined on all subsets of J. For the moment we assume these set functions are given by an oracle; it could conceivably be given by a simple formula as in the MNL model. Whenever we specify probabilities for a segment l for a given offer set S, we just write it with respect to $S_l := C_l \cap S$ (note that $P^l_j(S) = P^l_j(S_l)$). We define the vector $\vec{P}^l(S) = [P^l_1(S_l), \ldots, P^l_n(S_l)]$ (recall the no-purchase option is indexed by 0, so it is not included in this vector).

Given a customer arrival, and an offer set S, the firm sells $j \in S$ with probability $P_j(S) = \sum_l p_l P^l_j(S_l)$ and makes no sale with probability $P_0(S) = 1 - \sum_{j \in S} P_j(S)$. We define the vector $\vec{P}(S) = [P_1(S), \ldots, P_n(S)]$. Notice that $\vec{P}(S) = \sum_l p_l \vec{P}^l(S)$. We define the vectors $\vec{Q}^l(S) = A\vec{P}^l(S)$ and $\vec{Q}(S) = A\vec{P}(S)$. The revenue functions can be written as $R^l(S) = \sum_{j \in S_l} r_j P^l_j(S_l)$ and $R(S) = \sum_{j \in S} r_j P_j(S)$.

Let $V_t(\vec{c}_t)$ denote the maximum expected revenue that can be earned over the remaining time horizon $[t, T]$, given remaining capacity $\vec{c}_t$ in period t. Then $V_t(\vec{c}_t)$ must satisfy the well-known Bellman equation

$$V_t(\vec{c}_t) = \max_{S \subseteq J} \left\{ \sum_{j \in S} \lambda P_j(S) \Big(r_j + V_{t+1}(\vec{c}_t - \vec{A}_j) \Big) + \Big(\lambda P_0(S) + 1 - \lambda \Big) V_{t+1}(\vec{c}_t) \right\} \tag{1}$$

with the boundary condition $V_T(\vec{c}_T) = V_t(\vec{0}) = 0$ for all $\vec{c}_T$ and for all t. Let $V^{DP} = V_0(\vec{c}_0)$ denote the optimal value of this dynamic program from 0 to T, for the given initial capacity vector $\vec{c}_0$.

In our notation and demand model we broadly follow Bront et al. [2] and Liu and van Ryzin [7].

3 Approximations

We give the two formulations that we consider in this paper. Both can be considered as relaxations of a stochastic finite-time dynamic program that represents the network revenue management optimization problem.

3.1 Choice Deterministic Linear Program ($CDLP$)

The choice-based deterministic linear program ($CDLP$) defined in Gallego et al. [3] and Liu and van Ryzin [7] is as follows:

$$
(CDLP) \qquad \begin{aligned}
\max \quad & \sum_{S \subseteq J} \lambda R(S) w_S \\
\text{s.t.} \quad & \sum_{S \subseteq J} \lambda w_S \vec{Q}(S) \leq \vec{c}_0 \\
& \sum_{S \subseteq J} w_S = T \\
& 0 \leq w_S, \quad \forall\, S \subseteq J.
\end{aligned} \tag{2}
$$

The formulation has 2^n variables w_S that can be interpreted as the number of time periods each set is offered (including $w_{\{\emptyset\}}$). Liu and van Ryzin [7] show that the optimal objective value is an upper bound on V^{DP}. They also show that the problem can be solved efficiently by column-generation for the MNL model and non-overlapping segments. Bront et al. [2] and Rusmevichientong et al. [11] investigate this further and show that column generation is NP-hard whenever the consideration sets for the segments overlap, even for the MNL choice model.

3.2 Enhanced Segment-Based Deterministic Concave Program ($SDCP+$)

Talluri [12] proposed an upper bound on $CDLP$ called the segment-based deterministic concave program ($SDCP$) that coincides with the $CDLP$ when the segments do not overlap (formulation ($SDCP+$) given below without the equations (3)). In applications, the segments' consideration sets can overlap in a variety of ways, and the choice probabilities depend on the offer set, and need not follow any structure. Meissner et al. [9] develop a set of valid inequalities for $SDCP$ called product cuts as defined in (3) that tighten $SDCP$, and we call the combined formulation ($SDCP+$). The product cuts are *valid* in the sense that adding them to ($SDCP$) still results in an upper bound on the stochastic dynamic program (1).

$$
\begin{aligned}
& \max \quad \sum_{l=1}^{L} \sum_{S_l \subseteq C_l} \lambda_l R^l(S_l) w^l_{S_l} \\
& \text{s.t.} \\
(SDCP+) \qquad & \sum_{l=1}^{L} \vec{y_l} \le \vec{c}_0 \\
& \sum_{S_l \subseteq C_l} \lambda_l \vec{Q}^l(S_l) w^l_{S_l} \le \vec{y_l}, \quad \forall\, l \in \mathcal{L} \\
& \sum_{S_l \subseteq C_l} w^l_{S_l} = T, \quad \forall\, l \in \mathcal{L} \\
& \vec{y_l} \le \lambda_l T \vec{1}, \quad \forall\, l \in \mathcal{L} \\
& \sum_{\{S_l \subseteq C_l | S_l \supseteq S_{lm}\}} w^l_{S_l} - \sum_{\{S_m \subseteq C_m | S_m \supseteq S_{lm}\}} w^m_{S_m} = 0, \\
& \qquad \forall S_{lm} \subseteq C_l \cap C_m, \forall \{l, m\} \subset \mathcal{L} : C_l \cap C_m \neq \emptyset \qquad (3) \\
& w^l_{S_l} \ge 0, \qquad \forall S_l \subseteq C_l, \forall\, l \in \mathcal{L}, \\
& \vec{y_l} \ge \vec{0}, \qquad \forall\, l \in \mathcal{L}.
\end{aligned}
$$

The number of these cuts depends on the size of the overlap of consideration sets, and can be enumerated when overlaps are small. Even otherwise, one can add the cuts selectively, say restricting to the case where $|S_{lm}|$ is 2 or 3. The number of variables depends on the size of the consideration sets themselves, and can likewise be enumerated if consideration sets are small.

We state the following proposition whose proof is trivial:

Proposition 1. *If two segments have identical consideration sets, we can merge the two segments into one, changing $R^l(S_l)$ and $Q^l(S_l)$ as the revenue and probability for the two segments combined, without altering the solution of $(SDCP+)$.*

Therefore, whenever two segments have identical consideration sets, we merge them and consider a merged segment.

3.3 Example showing gap between $CDLP$ and $SDCP+$

Figure 1 shows an example with five products and three segments and their respective consideration sets. For this example we show that there is a gap between the optimal objective values of $SDCP+$ and $CDLP$. Assume $T = 1$, capacity $c = 1$, revenue $r_j = 1$ for all products $j \in J = \{1, 2, 3, 4, 5\}$, $\lambda_l = 1/3$ for all segments $l \in \{A, B, C\}$, and the purchase probabilities defined as follows: $P^A_j(\{1, 2\}) := 0.5$ for $j = 1, 2$, $P^A_j(\{2, 5\}) := 0.5$ for $j = 2, 5$, $P^B_2(\{2\}) := 1$, $P^B_j(\{1, 2, 3\}) := 1/3$ for $j = 1, 2, 3$, $P^C_4(\{4\}) := 1$ and $P^C_j(\{3, 5\}) := 0.5$ for $j = 3, 5$, and 0 for all other sets.

Claim: There is a feasible solution to $SDCP+$ for this network with objective value 1.
Proof
For this data, an optimal solution to $SDCP+$ is given by $y_l = 1/3$ for all segments l, and $w^A_{\{1,2\}} = w^A_{\{2,5\}} = 0.5$, $w^B_{\{2\}} = w^B_{\{1,2,3\}} = 0.5$, $w^C_{\{4\}} = w^C_{\{3,5\}} = 0.5$, and $w^l_{S_l} = 0$ otherwise for all $l \in \{A, B, C\}$, $S_l \subset C_l$. This

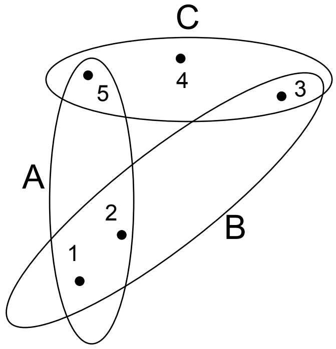

Figure 1: Consideration sets for 3 segments, 5 products

$\{l,m\}$	S_{lm}	$S_l \supseteq S_{lm}$	$\sum_{S_l \supseteq S_{lm}} w^l_{S_l}$	$S_m \supseteq S_{lm}$	$\sum_{S_m \supseteq S_{lm}} w^m_{S_m}$
{A,B}	{1}	{1},{1,2},{1,5},{1,2,5}	0.5	{1},{1,2},{1,3},{1,2,3}	0.5
{A,B}	{1,2}	{1,2},{1,2,5}	0.5	{1,2},{1,2,3}	0.5
{A,C}	{5}	{5},{1,5},{2,5},{1,2,5}	0.5	{5},{3,5},{4,5},{3,4,5}	0.5
{B,C}	{3}	{3},{1,3},{2,3},{1,2,3}	0.5	{3},{3,4},{3,5},{3,4,5}	0.5

Table 1: Evaluation of all product cuts (3) for the example in §3.3.

solution is feasible to ($SDCP+$) since $\lambda_l \sum_{S_l \subseteq C_l} Q^l(S_l) w^l_{S_l} = 1/3 = y_l$ for all segments l, the product cuts (3) for all pairs of segments $\{l,m\}$ and sets $S_{lm} \subset C_l \cap C_m$ are satisfied as reported in Table 1, and likewise the other constraints as can easily be checked. The objective value is 1 since $\lambda_l \sum_{S_l \subseteq C_l} R^l(S_l) w^l_{S_l} = 1/3$ for each $l \in \{A, B, C\}$.

□

Claim: There is no corresponding solution to $CDLP$ with the same objective value.

Proof

$CDLP$ has $2^5 = 32$ variables corresponding to subsets $S \subset J = \{1,2,3,4,5\}$. Under the single-leg example described above, we can enumerate all 32 subsets S and calculate the corresponding objective coefficient $\lambda R(S)$. We find that $\lambda R(S) \leq 2/3$ for all $S \subset J$, with equality reached for the sets $\{2,5\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{2,3,5\}$, $\{1,2,3,4\}$ and $\{1,2,3,5\}$. It follows that there can be no feasible solution to $CDLP$ that has objective value greater than 2/3 since the objective is a convex combination of these coefficients (note that $T = 1$). This proves the claim.

□

This simple example illustrates that there can be a positive gap between $CDLP$ and $SDCP+$ even if all product cuts are satisfied. The underlying reason for that gap is that there is no global offer set that can be projected onto the segment consideration sets so as to coincide with the segment-level optimal solution. These clashes cannot be removed by the product cuts because there is a cycle in the dependency structure of the consideration sets. We elaborate on this observation in the next section.

3.4 A tree structure for consideration sets

We seek to obtain a structural result on when $CDLP$ and $SDCP+$ are equivalent. To that end, let us define a bipartite *intersection graph of the consideration sets* as follows: There are two types of nodes, one type called segment node, the other is called intersection node. Each node of the former type corresponds to a segment, each of the latter represents a set of the form $C_k \cap C_l$ for some segment pair (k, l). If a set $S = C_k \cap C_l$ is the intersection of two distinct pairs $S = C_m \cap C_n$, then S is represented by a single node. Edges from segment k node connect to all the sets of the form $C_k \cap C_l \neq \emptyset$.

The example of §3.3 has a cycle as can be seen from Figure 2. This turns out to be the critical feature: If the segment consideration sets do not have a cycle, arranged say in the form of a tree (or, in general, a forest), then the product cuts are sufficient to ensure equivalence between *CDLP* and *SDCP+*.

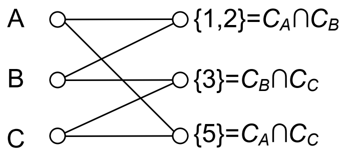

Figure 2: Intersection graph for the example in §3.3.

3.5 Equivalence for tree overlapping consideration sets

We show in this section that if the intersection graph is a forest then the *SDCP+* relaxation is tight. We use a merging procedure in the proof that for clarity we explain with a simple example shown in Figure 3 with two segments A, B and their consideration sets. Figure 3 shows a solution to *SDCP+* and we wish to construct a solution to *CDLP* from this solution. Note that the product cuts imply the restriction $w^A_{S_2} = w^B_{S_3}$, corresponding to the set U in the intersection of the consideration set. Moreover, $w^A_{S_1} + w^A_{S_2} = T$ and $w^B_{S_3} + w^B_{S_4} = T$. This implies that $w^A_{S_1} = w^B_{S_4}$. So we construct weights for the *CDLP* formulation as $w^{\text{CDLP}}_{S_1 \cup S_4} = w^A_{S_1} = w^B_{S_4}$ and $w^{\text{CDLP}}_{S_2 \cup S_3} = w^A_{S_2} = w^A_{S_3}$. This *CDLP* solution satisfies $w^{\text{CDLP}}_{S_1 \cup S_4} + w^{\text{CDLP}}_{S_2 \cup S_3} = T$, as well as the capacity constraints, and has the same objective value as the *SDCP+* solution.

The proof of Proposition 2 essentially repeats this argument for the much more complicated case with L segments and arbitrary consideration sets using an induction argument (made possible by the tree structure).

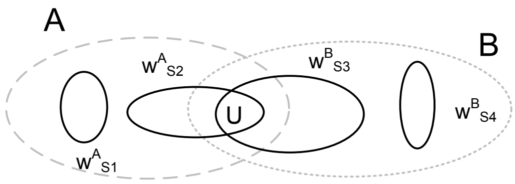

Figure 3: Merging procedure used in proof of Proposition 2.

Proposition 2. *If the intersection graph is a forest, then CDLP = SDCP+.*

Proof

Any solution to *CDLP* is a solution to *SDCP+* as shown in [9], hence *CDLP* $\leq$ *SDCP+*. It remains to show *CDLP* $\geq$ *SDCP+*. We construct a feasible solution to *CDLP* from a feasible solution to *SDCP+* by induction on the number of segments L.

Consider the case of a single segment $L = 1$, and a given feasible solution $(w^L_{S_L}, y_L)_{S_L}$ to *SDCP+*. Then $w^{CDLP}_S := w^L_S$ for all $S \subset J$ is a feasible solution to *CDLP* with the same objective value.

Next, we consider $L > 1$. Without loss of generality, the discussion will focus on a intersection graph that is a finite tree rather than a forest since the same arguments can be made for each tree that makes up the forest. Assuming that it is a tree, there must be at least two leaves, i.e. nodes with degree 1. By definition, intersection nodes have at least degree 2, so there exists a segment node that is a leaf. Without loss of generality, let this node correspond to the consideration set of segment L, and let $\overline{SDCP+}$ represent the problem $SDCP+$ with the segment L removed. Consider a feasible solution (w, y) to $SDCP+$, where (w, y) is shorthand notation for $w^l_{S_l}$ for $S_l \subseteq C_l$, for all segments $l \in \mathcal{L} := \{1, 2, \ldots, L\}$, and y_{il} for all resources i and $l \in \mathcal{L}$.

This solution induces a feasible solution $(\bar{w}, \bar{y})$ to $\overline{SDCP+}$ by defining $\bar{w}^l_{S_l} := w^l_{S_l}$ for all $S_l \subseteq C_l$, for all $l \in \bar{\mathcal{L}} := \mathcal{L} \setminus \{L\}$, and $\bar{y}_l := y_l$ for all $l \in \bar{\mathcal{L}}$. The solution $(\bar{w}, \bar{y})$ produces an objective value equal to that of $SDCP+$ less $\sum_{S_L \subseteq C_L} \lambda_L R^L(S_L) w^L_{S_L}$. By the induction assumption, there exists a feasible solution $\bar{w}^{CDLP}_S$ for all $S \subseteq \bar{J} := \cup_{l=1}^{L-1} C_l$ to $CDLP$ with the same objective value, and $\bar{w}^{CDLP}$ induces $(\bar{w}, \bar{y})$ meaning that $\bar{w}^l_{S_l} = \sum_{S \subseteq \bar{J} | S \cap C_l = S_l} \bar{w}^{CDLP}_S$ for all $l \in \bar{\mathcal{L}}$, $S_l \subseteq C_l$ for $l \in \bar{\mathcal{L}}$, and $\bar{y}_l = \sum_{S_l \subseteq C_l} \lambda_l \vec{Q}^l(S_l) \bar{w}^l_{S_l}$ for all $l \in \bar{\mathcal{L}}$.

Now we construct a feasible solution w^{CDLP}_S for all $S \subset J$ to $CDLP$ that induces (w, y) for $SDCP+$ with same objective value. Since L is a leaf of the intersection tree, all interactions with other segments are via a set S^{int} that is associated with the intersection node to which L is connected. Let us denote all segments that are connected to this intersection node by $\mathcal{L}^{\text{int}}$.

Consider a set $U \subseteq S^{\text{int}}$ that is *maximal* for segment L with respect to S^{int}, that is there is no set $S_L \subseteq C_L$ such that $U \subsetneq S_L \cap S^{\text{int}}$ and positive support $w^L_{S_L} > 0$. Note that for a feasible solution to $SDCP+$, the product cuts ensure that if a set is maximal for L with respect to S^{int}, it is maximal for all segments $l \in \mathcal{L}^{\text{int}}$ with respect to S^{int}. Moreover, from the definition of maximal, $\sum_{\{S_l \subseteq C_l | S_l \cap S^{\text{int}} \supseteq U\}} w^l_{S_l} = \sum_{\{S_l \subseteq C_l | S_l \cap S^{\text{int}} = U\}} w^l_{S_l}, \forall l \in \mathcal{L}^{\text{int}}$.

We select an arbitrary *maximal* set $U \subseteq S^{\text{int}}$ and segment $l \in \mathcal{L}^{\text{int}}$. The following argument shows that the total weight $\tau(U)$ that we offer sets that intersect with S^{int} exactly in U is the same in solutions w^L and $\bar{w}^{CDLP}$:

$$
\begin{aligned}
\tau(U) &= \sum_{S_L \subseteq C_L | S_L \cap S^{\text{int}} = U} w^L_{S_L} && (4)\\
&= \sum_{S_l \subseteq C_l | S_l \cap S^{\text{int}} = U} w^l_{S_l} && (5)\\
&= \sum_{S_l \subseteq C_l | S_l \cap S^{\text{int}} = U} \bar{w}^l_{S_l} && (6)\\
&= \sum_{S_l \subseteq C_l | S_l \cap S^{\text{int}} = U} \sum_{S \subseteq \bar{J} | S \cap C_l = S_l} \bar{w}^{CDLP}_S && (7)\\
&= \sum_{S \subseteq \bar{J} | S \cap S^{\text{int}} = U} \bar{w}^{CDLP}_S.
\end{aligned}
$$

The first equality holds by definition, the second due to maximality and the product cuts being satisfied by the solution w to $SDCP+$, the third since $w^l_{S_l} = \bar{w}^l_{S_l}$, the fourth because $\bar{w}^{CDLP}$ induces $\bar{w}$, and the final one as a result of a reformulation.

As a consequence, we can merge the solution $w^L_{S_l}$ with $\bar{w}^{CDLP}$ over total weight $\tau(U)$ to obtain w^{CDLP} for all sets that intersect with S^{int} only in the fixed set U. We illustrate the process in Figure 4 by drawing two parallel bars of equal length representing the weight $\tau(U)$, each bar with intervals corresponding to the support of the solutions w^L and $\bar{w}^{CDLP}$ (the order of the sets does not matter). Merging the sets as

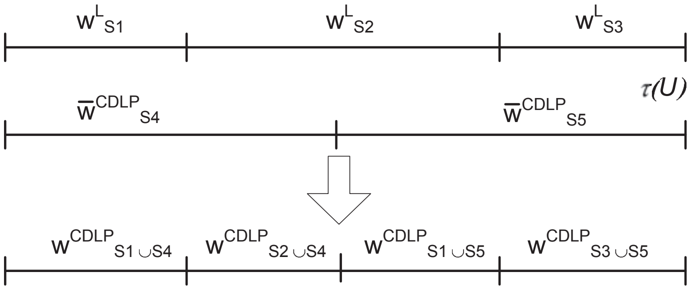

Figure 4: Illustration of the procedure to merge solutions in the support of w^L and $\bar{w}^{CDLP}$ to obtain w^{CDLP} for a fixed set $U \subseteq S^{\text{int}}$.

depicted ensures that the constructed solution w^{CDLP} induces w^L as well as w^l for $l \in \bar{\mathcal{L}}$ (the latter due to the induction assumption on $\bar{w}^{CDLP}$).

Now remove all the solution components $w^L_{S_L}$ and $w^l_{S_l}$ with positive support for all $l \in \mathcal{L}^{\text{int}}$ with $S_l \cap S^{\text{int}} = U$ and $S_L \cap S^{\text{int}} = U$. After the removal, the product cut equations for the remaining solution remain valid because of the equalities (4-5). We repeat this merging process by taking a maximal $U \subseteq S^{\text{int}}$ at each stage till we conclude with $U = \emptyset$. At every stage, as U is a maximal set, all the sets that contained U, namely sets of the form $U \subsetneq S_l \cap S^{\text{int}}, l \in \mathcal{L}^{\text{int}}$ were maximal sets in previous stages and therefore accounted for by equalities (4-5) for the set S_l; now combining it with the product cuts for the set U, we again obtain equalities (4-5).

The solution w^{CDLP} that emerges from this process is feasible to $CDLP$: it holds that $\sum_{U \subseteq S^{\text{int}}} \tau(U) = T$ (note that $U = \emptyset \subset S^{\text{int}}$), and therefore, by construction, $\sum_S w^{CDLP}_S = T$. That the capacity constraint of $CDLP$ is satisfied follows from the induction assumption that $\bar{w}^{CDLP}$ induces $(\bar{w}, \bar{y})$, with $\bar{w} := w$ and $\bar{y} := y$, combined with the fact that (w, y) is feasible to $SDCP+$ and that we constructed w^{CDLP} in a way such that we only added capacity consumption equal to that of segment L under solution w^L. So the combined solution also satisfies the induction for L segments.

The objective value of $CDLP$ equals that of $SDCP+$ because in the merging process we only add products of $C_L \setminus S^{\text{int}}$ to the solution $\bar{w}^{CDLP}$, and since these products do not influence other segments as they are only in the consideration set of segment L, we only add the contribution of segment L to the objective without a change of the contribution of other segments.

□

4 Numerical results

In this section we examine a test network used first in Liu and van Ryzin [7] and Bront et al. [2], and later in Talluri [12], Meissner et al. [9] and Meissner and Strauss [8]. For brevity, we just give the bare details of the network, specifically the consideration sets and the segments. We also give the value of a formulation called $SBLP+$ that was derived by Talluri [12]. $SBLP+$ applies the product-cuts to a compact formulation called sales-based linear program ($SBLP$) due to Gallego et al. [4], and it is interesting to observe that this formulation does not appear to give $CDLP$ value even with tree intersection structures.

4.1 Parallel flights example

The example consists of three parallel flight legs with 4 segments and 6 products, and the consideration sets and preference weights under the MNL choice model are given in Table 2. The intersection graph of the parallel flights example is shown in Figure 5 and it can be seen that it has a cycle. However, note that segments 3 and 4 have identical consideration sets, and by Proposition 1 we merge them and obtain a tree intersection graph. This explains why the $SDCP+$ values are identical to that of $CDLP$ in Table 3.

Segment	Consideration set	Pref. vector	λ_l	Description
1	{2,4,6}	[5,10,1]	0.1	Price insensitive, afternoon preference
2	{1,3,5}	[5,1,10]	0.15	Price sensitive, evening preference
3	{1,2,3,4,5,6}	[10,8,6,4,3,1]	0.2	Early preference, price sensitive
4	{1,2,3,4,5,6}	[8,10,4,6,1,3]	0.05	Price insensitive, early preference

Table 2: Segment definitions for Parallel Flights Example.

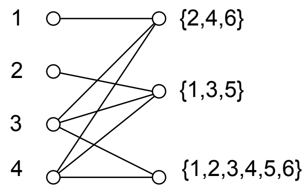

Figure 5: Intersection graph for the parallel-flights example

The example also explains the gap between the objective values of the $SDCP+$ and the $SBLP+$ formulations. Since $SDCP+$ applies to any general discrete-choice model, we are able to use Proposition 1 and consider a merged segment with a resulting tree intersection structure. One cannot do that for the $SBLP+$ formulation as it is specific to the MNL model and merging the segments would give a mixture model, so the cycle in the intersection graph persists for the $SBLP+$ formulation.

4.2 Upper Bounds on $CDLP$

We showed in §3.3 that there can be a gap between $CDLP$ and $SDCP+$ when there is a cycle in the intersection graph. For the Parallel Flights example there is also a cycle, and there is a (small) gap between $SBLP+$ and $CDLP$. It follows that $SDCP+$ is in general not equivalent to $SBLP+$ so that the more complex formulation of $SDCP+$ can result in improved bounds under the MNL choice model. In contrast, $SDCP+ = CDLP$ as predicted by our result on the structure of the consideration sets.

5 Conclusions

Bounds on the stochastic dynamic program that represents the network revenue management problem with customer choice behavior are used to obtain good control policies. The $CDLP$ formulation gives such a bound, but is computationally intractable for realistically sized applications when segment consideration sets overlap. Sales-based Linear Program ($SBLP$) and the Segment-based Deterministic Concave Program ($SDCP$) are two weaker tractable approximations that can be tightened ($SBLP+$ and $SDCP+$), while

α	v_0	$CDLP$	$SDCP+$	$SDCP$	$SBLP+$
0.6	[1,5,5,1]	56,884	56,884	58,755	56,912
	[1,10,5,1]	56,848	56,848	58,755	56,884
	[5,20,10,5]	53,820	53,820	54,684	53,842
0.8	[1,5,5,1]	71,936	71,936	73,870	72,031
	[1,10,5,1]	71,795	71,795	73,870	71,936
	[5,20,10,5]	61,868	61,868	63,439	61,996
1	[1,5,5,1]	79,156	79,156	85,424	80,078
	[1,10,5,1]	76,866	76,866	83,376	77,605
	[5,20,10,5]	63,256	63,256	65,847	63,274
1.2	[1,5,5,1]	80,371	80,371	88,331	81,003
	[1,10,5,1]	78,045	78,045	86,332	78,385
	[5,20,10,5]	63,296	63,296	66,647	63,321

Table 3: Upper bounds for Parallel Flights/overlapping segments example (Bront et al. [2]). Capacities are scaled by multiplying the capacities by a factor α. Different no-purchase weights are given in v_0, using the same choices as in Liu and van Ryzin [7] and Bront et al. [2]. $SDCP+$ was obtained using product cuts corresponding to subsets $|S_{lm}| \leq 2$

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still maintaining tractability, by adding valid inequalities. Their performance on test problems has been outstanding. Naturally, this raises the question on what conditions can guarantee equivalence of the formulations.

In this paper, we obtain a structural result to this end, namely that $CDLP$ and $SDCP+$ are equivalent if the intersection graph of the segment consideration sets is a tree (or a forest). This implies that these efficient solution methods will be guaranteed to yield the same control policies as $CDLP$ if demand can be modeled without cycles in the overlap structure of the segment consideration sets.

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