Elasticity Determinants of Inequality Reducing Income Taxation

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Appendix

In this appendix, we present the proofs of Theorem 2, Theorem 3, Proposition 1, and Proposition 2. Each proof is preceded by a restatement of its corresponding theorem for the convenience of the reader.

The proofs of Theorem 2 and that of Theorem 3 adapt arguments from the proofs of Theorem 2 and Theorem 3 in Carbonell-Nicolau and Llavador (2018).

The following two lemmas, whose proofs can be found in Carbonell-Nicolau and Llavador (2018) (see their Lemma 2 and Lemma 3), are instrumental in the proofs of Theorem 2 and Theorem 3.

Lemma 1. *Given $u \in \mathcal{U}$, $(c, y) \in \mathbb{R}_{++}^2$, and $q \in (0, +\infty)$, there exists an $a > y$ such that $\eta^a(c, y) = q$.*

Lemma 2. *Given $u \in \mathcal{U}^*$, a tax schedule $T \in \mathcal{T}$ is u -iir if and only if for any ability distribution $\mathbf{a} \in \mathcal{A}$ and for any pre-tax and post-tax income functions y^u and x^u ,*

$$\frac{x^u(a_i, T)}{y^u(a_i, 0)} \geq \frac{x^u(a_{i+1}, T)}{y^u(a_{i+1}, 0)} \quad \forall i \in \{1, \dots, n-1\} : y^u(a_i, 0) > 0. \quad (17)$$

A Proof of Theorem 2

Theorem 2. *Given $u \in \mathcal{U}^*$ and $(B, (R_k)) \in \mathcal{D}$,*

$$[\mathcal{T}_{u\text{-iir}} \subseteq \mathcal{T}_{\text{prog}}] \quad \text{and} \quad \left[\mathcal{T}_{\text{prog}}(\bar{B}, (R_k)) \subseteq \mathcal{T}_{u\text{-iir}} \iff \mathcal{T}_{\text{lin}}\left(\bar{B}, \bigcup_k R_k\right) \subseteq \mathcal{T}_{u\text{-iir}} \right].$$

Proof. The first containment follows immediately from Theorem 1.

Suppose that $u \in \mathcal{U}^*$ and $(B, (R_k)) \in \mathcal{D}$.

Since $\mathcal{T}_{\text{lin}}(\bar{B}, \bigcup_k R_k) \subseteq \mathcal{T}_{\text{prog}}(\bar{B}, (R_k))$, the ‘only if’ part of the equivalence is obvious.

Assume now that the members of $\mathcal{T}_{\text{lin}}(\bar{B}, \bigcup_k R_k) \subseteq \mathcal{T}_{u\text{-iir}}$. We need to prove that $\mathcal{T}_{\text{prog}}(\bar{B}, (R_k)) \subseteq \mathcal{T}_{u\text{-iir}}$. By Lemma 2, this is equivalent to showing that that condition (17) holds for any $T \in \mathcal{T}_{\text{prog}}(\bar{B}, (R_k))$, for any ability distribution $\mathbf{a} \in \mathcal{A}$, and for any pre-tax and post-tax income functions y^u and x^u .

Take $T = (\alpha_0, \mathbf{t}, \bar{\mathbf{y}}) \in \mathcal{T}_{\text{prog}}(\bar{B}, (R_k))$ and, for each income threshold $\bar{y}_k$ of T , define the linear tax schedule $T_k(y) := t_k y - \alpha_k$ with $\alpha_0 \in \bar{B}$, $t_k \in R_k$ for $k \in \{0, \dots, K\}$, and $\alpha_k := \alpha_{k-1} + (t_k - t_{k-1})\bar{y}_k$ for $k \in \{1, \dots, K\}$.

Pre-tax and post-tax income functions, y^u and x^u , are uniquely defined, since preferences are strictly quasiconcave and the tax function T is convex. For $k \in \{1, \dots, K\}$, define the abilities a_k^- and a_k such that

$$a_k^- := \min \{a : y^u(a, T_{k-1}) = \bar{y}_k^u\} \quad \text{and} \quad a_k := \max \{a : y^u(a, T_k) = \bar{y}_k^u\}$$

Figure 3: Figure for Theorem 2

(see Figure 3). Lemma 1 guarantees that a_k^- and a_k exist and are well-defined for all $k \in \{1, \dots, K\}$.

Furthermore, since T is marginal-rate progressive (and hence $t_{k-1} < t_k$ for all $k \in \{1, \dots, K\}$), agent monotonicity (Definition 4) implies that $a_k^- \leq a_k < a_{k+1}^-$.

Next, define the following family of sets covering $(0, +\infty)$:

$$\mathfrak{A} := \left\{ (0, a_1^-], \{[a_k^-, a_k]\}_{k=1}^K, \{[a_k, a_{k+1}^-]\}_{k=1}^{K-1}, [a_K, +\infty) \right\}.$$

We first show that condition (17) is satisfied for ability distributions contained in each element of the family $\mathfrak{A}$.

(i) Consider first the interval $(0, a_1^-]$. Observe that $y^u(a, T) = y^u(a, T_0)$ for all $a \leq a_1^-$.

Because T_0 is a linear tax, it is *u-irr*, and so Lemma 2 gives

$$\frac{x^u(a, T)}{y^u(a, 0)} = \frac{x^u(a, T_0)}{y^u(a, 0)} \geq \frac{x^u(a', T_0)}{y^u(a', 0)} = \frac{x^u(a', T)}{y^u(a', 0)} \quad \forall a \leq a' \leq a_1^-. \quad (18)$$

(ii) For $[a_K, +\infty)$, a symmetric argument shows that

$$\frac{x^u(a, T)}{y^u(a, 0)} \geq \frac{x^u(a', T)}{y^u(a', 0)} \quad \forall a_K \leq a \leq a'. \quad (19)$$

(iii) Now consider the interval $[a_k^-, a_k]$ for $k \in \{1, \dots, K\}$. Observe that

$$y^u(a_k, T) = y^u(a_k, T_k) = \bar{y}_k = y^u(a_k^-, T_{k-1}) = y^u(a_k^-, T).$$

Because the map $a \mapsto y^u(a, T)$ is monotone (Mirrlees, 1971, Theorem 1), $y^u(a, T) = \bar{y}_k$ for all $a \in [a_k^-, a_k]$, and $y^u(a', 0) \geq y^u(a, 0)$ for all $a_k^- \leq a \leq a' \leq a_k$. Therefore,

$$\frac{x^u(a, T)}{y^u(a, 0)} = \frac{\bar{y}_k - T(\bar{y}_k)}{y^u(a, 0)} \geq \frac{\bar{y}_k - T(\bar{y}_k)}{y^u(a', 0)} = \frac{x^u(a', T)}{y^u(a', 0)} \quad \forall a, a' \in [a_k^-, a_k], a \leq a'. \quad (20)$$

(iv) Finally, consider the interval $[a_k, a_{k+1}^-]$ for $k \in \{1, \dots, K-1\}$. By construction, we have $y^u(a, T) = y^u(a, T_k)$ for all $a \in [a_k, a_{k+1}^-]$. Therefore, since T_k is a linear tax in $\mathcal{T}_{lin}(\bar{B}, \cup R_k)$, and hence u -iir, Lemma 2 gives

$$\frac{x^u(a, T)}{y^u(a, 0)} = \frac{x^u(a, T_k)}{y^u(a, 0)} \geq \frac{x^u(a', T_k)}{y^u(a', 0)} = \frac{x^u(a', T)}{y^u(a', 0)} \quad \forall a, a' \in [a_k, a_{k+1}^-], a \leq a'. \quad (21)$$

Combining equations (18)-(21) we obtain (17) for every $\mathbf{a} \in \mathcal{A}$. ■

B Proof of Theorem 3

Theorem 3. For $u \in \mathcal{U}^*$, the members of $\mathcal{T}_{lin}(B, R)$ are u -iir if and only if $u \in \mathcal{U}(B, R)$.

Proof. Given $u \in \mathcal{U}^*$, $B \subseteq \mathbb{R}_+$, and $R \subseteq [0, 1]$, the members $T(y) = -b + ry$ of $\mathcal{T}_{lin}(B, R)$ are u -iir if and only if the map

$$a \mapsto \frac{x^u(a, T)}{y^u(a, 0)} = \frac{a(1-r)l^u(a, T) + b}{al^u(a, 0)} = \frac{a(1-r)l^u((1-r)a, b) + b}{al^u(a, 0)} \quad (22)$$

defined on $\mathbb{R}_{++}$ is nonincreasing for every $(b, r) \in B \times R$ (Lemma 2). Equivalently, the members of $\mathcal{T}_{lin}(B, R)$ are u -iir if and only if

$$\begin{aligned} & \frac{(1-r) \left((1-r)a' \frac{\partial l^u((1-r)a', b)}{\partial a} + l^u((1-r)a', b) \right) a' l^u(a', 0)}{(a' l^u(a', 0))^2} \\ & - \frac{((1-r)a' l^u((1-r)a', b) + b) \left(a' \frac{\partial l^u(a', 0)}{\partial a} + l^u(a', 0) \right)}{(a' l^u(a', 0))^2} \leq 0 \end{aligned} \quad (23)$$

for every $(a', b, r) \in \mathbb{R}_{++} \times B \times R$.¹⁶ Since the above inequality can be expressed as

$$\frac{(1-r)a' \left((1-r)a' \frac{\partial l^u((1-r)a', b)}{\partial a} + l^u((1-r)a', b) \right)}{(1-r)a' l^u((1-r)a', b) + b} \leq \frac{a' \left(a' \frac{\partial l^u(a', 0)}{\partial a} + l^u(a', 0) \right)}{a' l^u(a', 0)},$$

or, equivalently, as

$$\zeta^u((1-r)a', b) \leq \zeta^u(a', 0), \quad (24)$$

¹⁶More precisely, the map defined in (22) is nonincreasing for every $(b, r) \in B \times R$ if and only if for every $(b, r) \in B \times R$, (23) holds for all but perhaps one $a' > 0$.

we see that the members of $\mathcal{T}_{lin}(B, R)$ are u -iir if and only if (24) holds for every $(a', b, r) \in \mathbb{R}_{++} \times B \times R$. Consequently, for $u \in \mathcal{U}^*$, the members of $\mathcal{T}_{lin}(B, R)$ are u -iir if and only if $u \in \mathcal{U}(B, R)$. ■

C Proof of Proposition 1

Proposition 1. *Let u be the CES utility function given in (10). Suppose that $R \subseteq [0, 1]$ and $\sup R < 1$. Then there exists $\underline{b} \geq 0$ such that $\mathcal{T}_{prog}(B^*, R) \subseteq \mathcal{T}_{u-iir} \subseteq \mathcal{T}_{prog}$ if and only if $\gamma \in [\frac{1}{2}, 1)$, where $B^* := \{b \in \mathbb{R}_+ : b \geq \underline{b}\}$.*

Proof. Since $u \in \mathcal{U}^*$, given $\underline{b} \geq 0$, $B^* := \{b \in \mathbb{R}_+ : b \geq \underline{b}\}$, and $R \subseteq [0, 1]$, Theorem 4 gives $\mathcal{T}_{prog}(B^*, R) \subseteq \mathcal{T}_{u-iir} \subseteq \mathcal{T}_{prog}$ if and only if $u \in \mathcal{U}(B^*, R)$.

Given $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$, if $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} < b$, then $(\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} < b$, implying $\zeta^{CES}(a, 0) = 0 \geq 0 = \zeta^{CES}((1-r)a, b)$. If $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} \geq b > (\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}}$, then $\zeta^{CES}(a, 0) \geq 0 = \zeta^{CES}((1-r)a, b)$. Let $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} \geq (\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} \geq b$, and split the total effect into the subsidy and tax-rate effects:

$$\zeta^{CES}((1-r)a, b) - \zeta^{CES}(a, 0) = \left[\zeta^{CES}((1-r)a, b) - \zeta^{CES}((1-r)a, 0) \right] + \left[\zeta^{CES}((1-r)a, 0) - \zeta^{CES}(a, 0) \right].$$

After some manipulation, it follows that the subsidy effect is non-positive:

$$\zeta^{CES}((1-r)a, b) - \zeta^{CES}((1-r)a, 0) = \frac{-b}{(1-r)a + b} \leq 0; \quad (25)$$

while the tax-rate effect is non-negative:

$$\zeta^{CES}((1-r)a, 0) - \zeta^{CES}(a, 0) = \frac{\gamma \left(a^{\frac{\gamma}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \left(1 - (1-r)^{\frac{\gamma}{1-\gamma}} \right) \right)}{(1-\gamma) \left(\left(a^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \left(((1-r)a)^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \right)} \geq 0 \quad (26)$$

Therefore, the total effect is negative if and only if

$$\zeta^{CES}((1-r)a, b) - \zeta^{CES}(a, 0) = \frac{\gamma \left(a^{\frac{\gamma}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \left(1 - (1-r)^{\frac{\gamma}{1-\gamma}} \right) \right)}{(1-\gamma) \left(\left(a^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \left(((1-r)a)^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \right)} - \frac{b}{a + b} \leq 0$$

The previous inequality is equivalent to

$$\frac{\gamma \left(a^{\frac{1}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \left(1 - (1-r)^{\frac{\gamma}{1-\gamma}} \right) \right)}{(1-\gamma) \left(\left(a^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \left(((1-r)a)^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \right)} \leq b \left(1 - \frac{\gamma \left(a^{\frac{\gamma}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \left(1 - (1-r)^{\frac{\gamma}{1-\gamma}} \right) \right)}{(1-\gamma) \left(\left(a^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \left(((1-r)a)^{\frac{\gamma}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \right) \right)} \right)$$

Arranging terms gives

$$\begin{aligned}
& a^{\frac{1}{1-\gamma}}(a(1-r))^{\frac{1}{1-\gamma}} \left[(a(1-r))^{-1}((1-\gamma)a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}})(a(1-r)\beta^{\frac{1}{1-\gamma}} + (a(1-r))^{\frac{1}{1-\gamma}}) \right. \\
& \quad \left. - \gamma\beta^{\frac{1}{1-\gamma}}(a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}}) \right] b \\
& \geq a^{\frac{1}{1-\gamma}}(a(1-r))^{\frac{1}{1-\gamma}} \left[(a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}})(a(1-r)\beta^{\frac{1}{1-\gamma}} + (1-\gamma)(a(1-r))^{\frac{1}{1-\gamma}}) \right. \\
& \quad \left. - ((1-\gamma)a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}})(a(1-r)\beta^{\frac{1}{1-\gamma}} + (a(1-r))^{\frac{1}{1-\gamma}}) \right].
\end{aligned}$$

This simplifies to

$$\begin{aligned}
& \left[((1-\gamma)a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}}) \left(\beta^{\frac{1}{1-\gamma}} + (a(1-r))^{\frac{\gamma}{1-\gamma}} \right) - \gamma\beta^{\frac{1}{1-\gamma}} \left(a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}} \right) \right] b \\
& \geq a^{1+\frac{1}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \gamma \left(1-r - (1-r)^{\frac{1}{1-\gamma}} \right). \tag{27}
\end{aligned}$$

We claim that if $\sup R < 1$ there exists $\underline{b} \geq 0$ such that $\mathcal{T}_{prog}(B^*, R) \subseteq \mathcal{T}_{u-iir} \subseteq \mathcal{T}_{prog}$ if and only if $\gamma \in [\frac{1}{2}, 1)$. To see this, it suffices to show that (i) for $\gamma < \frac{1}{2}$ and $\underline{b} \geq 0$, there exists $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} \geq (\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} \geq \underline{b}$ such that (27) does not hold, and (ii) for $\gamma \in [\frac{1}{2}, 1)$, there exists $\underline{b} \geq 0$ such that for $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} \geq (\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} \geq \underline{b}$, (27) holds.

The bracketed term on the left-hand side of (27) is positive for a large enough (indeed, for any a if $\gamma < 0$ or $\gamma = 1/2$). Therefore, for a large enough, (27) is equivalent to

$$b \geq \frac{a^{1+\frac{1}{1-\gamma}} \beta^{\frac{1}{1-\gamma}} \gamma \left(1-r - (1-r)^{\frac{1}{1-\gamma}} \right)}{\left((1-\gamma)a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}} \right) \left(\beta^{\frac{1}{1-\gamma}} + (a(1-r))^{\frac{\gamma}{1-\gamma}} \right) - \gamma\beta^{\frac{1}{1-\gamma}} \left(a^{\frac{1}{1-\gamma}} + a\beta^{\frac{1}{1-\gamma}} \right)} =: \Lambda(a).$$

For $\gamma < 1/2, \gamma \neq 0$, $\Lambda(a)$ converges to infinity as a tends to infinity. Consequently, for any $\underline{b} \geq 0$, there exists $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(\frac{a}{\beta})^{\frac{1}{1-\gamma}} \geq (\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} \geq \underline{b}$ such that (27) does not hold.

If $\gamma = \frac{1}{2}$, $\Lambda(a)$ is increasing in a and converges to $r\beta^2$ as $a \rightarrow \infty$. Consequently, it suffices to set $\underline{b} = \beta^2 \sup R \leq \beta^2$.¹⁷

If $\gamma \in (\frac{1}{2}, 1)$, $\Lambda(a)$ tends to 0 as $a \rightarrow \infty$, and so, if $\sup R < 1$, there exists $\underline{b}$ such that for $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} \geq \underline{b}$, (27) holds.¹⁸ ■

¹⁷For example, if $\beta = 1$ and the maximum marginal tax rate is $\frac{1}{2}$, then it suffices to consider the set of all the marginal-rate progressive tax schedules that provide a subsidy of at least $\frac{1}{2}$.

¹⁸For instance, if $\gamma = \frac{3}{4}$, $\sup R = 0.6$, and $\beta = 1$, (27) becomes

$$[(0.25a^4 + a)(1 + (a(1-r))^3) - 0.75(a^4 + a)] b \geq 0.75a^4(1-r)(1 - (1-r)^3). \tag{28}$$

If $\underline{b} = 81$, then given $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(\frac{(1-r)a}{\beta})^{\frac{1}{1-\gamma}} = (1-r)^4 a^4 \geq \underline{b} \geq \underline{b}$, i.e., $a \geq \frac{\underline{b}^{\frac{1}{4}}}{1-r} \geq \frac{\underline{b}^{\frac{1}{4}}}{1-r} = \frac{3}{1-r}$, (28) is equivalent to

$$b \geq \frac{0.75a^4(1-r)(1 - (1-r)^3)}{(0.25a^4 + a)(1 + (a(1-r))^3) - 0.75(a^4 + a)}, \tag{29}$$

D Proof of Proposition 2

Proposition 2. *Let u be the quasi-linear utility function given in (13). Suppose that $R \subseteq [0, 1)$ and $\sup R < 1$. Then there exists $\underline{b} \geq 0$ such that $\mathcal{T}_{\text{prog}}(B^*, R) \subseteq \mathcal{T}_{u\text{-iir}} \subseteq \mathcal{T}_{\text{prog}}$ if and only if $\delta \in (0, 1)$, where $B^* := \{b \in \mathbb{R}_+ : b \geq \underline{b}\}$.*

Proof. Since $u \in \mathcal{U}^*$, given $\underline{b} \geq 0$, $B^* := \{b \in \mathbb{R}_+ : b \geq \underline{b}\}$, and $R \subseteq [0, 1)$, Theorem 4 gives $\mathcal{T}_{\text{prog}}(B^*, R) \subseteq \mathcal{T}_{u\text{-iir}} \subseteq \mathcal{T}_{\text{prog}}$ if and only if $u \in \mathcal{U}(B^*, R)$.

Given $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$, if $a \leq \beta/(1-r)$, then $\zeta^u((1-r)a, b) = 0 \leq \zeta^u(a, 0)$. If $(1-r)a > \beta$, then $\zeta^{QL}(a, 0) \geq \zeta^{QL}((1-r)a, b)$ if and only if

$$\frac{(1-r)a(\theta((1-r)a) - \theta(a)) - b((1-\delta)\theta(a) + \delta)}{\delta(1-\theta(a))((1-r)a(1-\theta((1-r)a)) + b)} \leq 0, \quad (30)$$

where $\theta(a) = \left(\frac{\beta}{a}\right)^{1/\delta}$.

Observe that $\theta(a) < 1$ since $a > \beta$; and that $\theta((1-r)a) - \theta(a) = \left(\left(\frac{1}{1-r}\right)^{1/\delta} - 1\right)\left(\frac{\beta}{a}\right)^{1/\delta} > 0$, for $r \in (0, 1)$. Therefore, the denominator in (30) is positive, and the inequality holds if and only if the numerator is negative, that is,

$$(1-r)a(\theta((1-r)a) - \theta(a)) - b((1-\delta)\theta(a) + \delta) \leq 0.$$

After some manipulation, this is equivalent to

$$b \geq \frac{a\beta^{1/\delta}(1-(1-r)^{1/\delta})(1-r)^{\frac{\delta-1}{\delta}}}{\delta a^{1/\delta} + (1-\delta)\beta^{1/\delta}}. \quad (31)$$

The right-hand side of the inequality is bounded above if and only if $\delta \leq 1$ (otherwise it tends to infinity as $a \rightarrow \infty$). Hence, for $\delta \in (0, 1)$, there exists $\underline{b}$ such that for $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(1-r)a > \beta$, (31) holds. And, if $\delta > 1$ and $\underline{b} \geq 0$, there exists $(a, b, r) \in \mathbb{R}_{++} \times B^* \times R$ with $(1-r)a > \beta$ such that (31) does not hold. $\blacksquare$

and since

$$\frac{0.75a^4(1-r)(1-(1-r)^3)}{(0.25a^4 + a)(1 + (a(1-r))^3) - 0.75(a^4 + a)} < 81$$

and $b \geq \underline{b}$, it follows that (29) holds.